

Algebras of distributions for binary semi-isolating formulas of a complete theory*

Sergey V. Sudoplatov[†]

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Abstract

We define a class of algebras describing links of binary semi-isolating formulas on a set of realizations for a family of 1-types of a complete theory. These algebras include algebras of isolating formulas considered before. We prove that a set of labels for binary semi-isolating formulas on a set of realizations for a 1-type p forms a monoid of a special form with a partial order inducing ranks for labels, with set-theoretic operations, and with a composition. We describe the class of these structures. A description of the class of structures relative to families of 1-types is given.

Key words: type, complete theory, algebra of binary semi-isolating formulas, join of monoids, deterministic structure.

In [1], a series of constructions is introduced admitting to realize key properties of countable theories and to obtain a classification of countable models of small (in particular, of Ehrenfeucht) theories with respect to two basic characteristics: Rudin–Keisler preorders and distribution functions for numbers of limit models. The construction of these theories is essentially based on the definition of special directed graphs with colored vertices and arcs as well as on the definition of $(n + 1)$ -ary predicates that turn prime models

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[†]sudoplat@math.nsc.ru

over realizations of n -types to prime models over realizations of 1-types and reducing links between prime models over finite sets to links between prime models over elements such that these links are given by definable sets of arcs and edges.

In the paper, we develop a general approach to the description of binary links between realizations of 1-types in terms of labels of pairwise non-equivalent isolating formulas [2] to sets of labels of semi-isolating formulas.

We use the standard relation algebraic, model-theoretical, semigroup, and graph-theoretic terminology [3]–[14] as well as some notions, notations, and constructions in [1, 2].

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1. Preliminary notions, notations, and properties

Definition [1, 2, 15, 16]. Let T be a complete theory, $\mathcal{M} \models T$. Consider types $p(x), q(y) \in S(\emptyset)$, realized in \mathcal{M} , and all (p, q) -*preserving* (p, q) -*semi-isolating*, $(p \rightarrow q)$ -, or $(q \leftarrow p)$ -*formulas* $\varphi(x, y)$ of T , i. e., formulas for which there is $a \in M$ such that $\models p(a)$ and $\varphi(a, y) \vdash q(y)$. Now, for each such a formula $\varphi(x, y)$, we define a binary relation $R_{p, \varphi, q} \equiv \{(a, b) \mid \mathcal{M} \models p(a) \wedge \varphi(a, b)\}$. If $(a, b) \in R_{p, \varphi, q}$, (a, b) is called a (p, φ, q) -*arc*. If $\varphi(a, y)$ is principal (over a), the (p, φ, q) -arc (a, b) is also *principal*.

If, in addition, $\varphi(x, y)$ is a $(p \leftrightarrow q)$ -*formula*, i. e., it is both a $(p \rightarrow q)$ - and a $(q \rightarrow p)$ -formula then the set $[a, b] \equiv \{(a, b), (b, a)\}$ is said to be a (p, φ, q) -*edge*. If the (p, φ, q) -edge $[a, b]$ consists of principal (p, φ, q) - and $(q, \varphi(y, x), p)$ -arcs then $[a, b]$ is a *principal* (p, φ, q) -edge.

(p, φ, q) -arcs and (p, φ, q) -edges are called *arcs* and *edges* respectively if we say about fixed or some $(p \rightarrow q)$ -formula $\varphi(x, y)$. If (a, b) is a principal (p, φ, q) -arc such that the pair (b, a) is not a principal arc (on any formula), that is $(b, a) \notin \text{SI}_{\{p, q\}}$, then (a, b) is called *irreversible*. If (a, b) is a (p, φ, q) -arc and (b, a) is not a (q, φ, p) -arc then (a, b) is also an *irreversible* arc.

For types $p(x), q(y) \in S(\emptyset)$, we denote by $\text{SICF}(p, q)$ the set of $(p \rightarrow q)$ -formulas $\varphi(x, y)$ such that $\{\varphi(a, y)\}$ is consistent for $\models p(a)$. Let $\text{SICE}(p, q)$ be the set of pairs of formulas $(\varphi(x, y), \psi(x, y)) \in \text{SICF}(p, q)$ such that for any (some) realization a of p the sets of solutions for $\varphi(a, y)$ and $\psi(a, y)$ coincide. Clearly, $\text{SICE}(p, q)$ is an equivalence relation on the set $\text{SICF}(p, q)$. Notice that each $\text{SICE}(p, q)$ -class E corresponds to either a set of (p, φ, q) -edges, or a set of irreversible (p, φ, q) -arcs, or simultaneously a set of (p, φ, q) -edges and of irreversible (p, φ, q) -arcs linking realizations of p and q by any (some)

formula φ in E . Thus the quotient $\text{SICF}(p, q)/\text{SICE}(p, q)$ is represented as a disjoint union of sets $\text{SICFE}(p, q)$, $\text{SICFA}(p, q)$, and $\text{SICFM}(p, q)$, where $\text{SICFE}(p, q)$ consists of $\text{SICE}(p, q)$ -classes correspondent to sets of edges, $\text{SICFA}(p, q)$ consists of $\text{SICE}(p, q)$ -classes correspondent to sets of irreversible arcs, and $\text{SICFM}(p, q)$ consists of $\text{SICE}(p, q)$ -classes correspondent to sets containing edges and irreversible arcs.

The sets $\text{SICF}(p, p)$, $\text{SICE}(p, p)$, $\text{SICFE}(p, p)$, $\text{SICFA}(p, p)$, and $\text{SICFM}(p, p)$ are denoted by $\text{SICF}(p)$, $\text{SICE}(p)$, $\text{SICFE}(p)$, $\text{SICFA}(p)$, and $\text{SICFM}(p)$ respectively.

Let T be a complete theory without finite models, $U = U^- \dot{\cup} \{0\} \dot{\cup} U^+ \dot{\cup} U'$ be an alphabet of cardinality $\geq |S(T)|$ and consisting of *negative elements* $u^- \in U^-$, *positive elements* $u^+ \in U^+$, *neutral elements* $u' \in U'$, and zero 0. As usual, we write $u < 0$ for any $u \in U^-$ and $u > 0$ for any $u \in U^+$. The set $U^- \cup \{0\}$ is denoted by $U^{\leq 0}$ and $U^+ \cup \{0\}$ is denoted by $U^{\geq 0}$. Elements of U are called *labels*.

Let $\nu(p, q): \text{SICF}(p, q)/\text{SICE}(p, q) \rightarrow U$ be injective *labelling functions*, $p(x), q(y) \in S(\emptyset)$, for which negative elements correspond to the classes in $\text{SICFA}(p, q)/\text{SICE}(p, q)$, positive elements and 0 correspond to the classes in $\text{SICFE}(p, q)/\text{SICE}(p, q)$ such that 0 is defined only for $p = q$ and is represented by the formula $(x \approx y)$, and neutral elements code the classes in $\text{SICFM}(p, q)/\text{SICE}(p, q)$, $\nu(p) = \nu(p, p)$. We additionally assume that $\rho_{\nu(p)} \cap \rho_{\nu(q)} = \{0\}$ for $p \neq q$ where, as usual, we denote by ρ_f the image of the function f and $\rho_{\nu(p, q)} \cap \rho_{\nu(p', q')} = \emptyset$ if $p \neq q$ and $(p, q) \neq (p', q')$. Labelling functions with the properties above as well families of these functions are said to be *regular*. Further we shall consider only regular labelling functions and their regular families.

The labels, correspondent to isolating formulas, are said to be *isolating* whereas each label in $\bigcup_{p, q \in S^1(\emptyset)} \rho_{\nu(p, q)}$ is *semi-isolating*. By the definition, each isolating label belongs to $U^- \dot{\cup} \{0\} \dot{\cup} U^+$, i. e., it is not neutral.

We denote by $\theta_{p, u, q}(x, y)$ formulas in $\text{SICF}(p, q)$ with a label $u \in \rho_{\nu(p, q)}$. If the type p is fixed and $p = q$ then the formula $\theta_{p, u, q}(x, y)$ is denoted by $\theta_u(x, y)$.

Note that if $\theta_{p, u, q}(x, y)$ and $\theta_{q, v, p}(x, y)$ are formulas witnessing that for realizations a and b of p and q respectively the pairs (a, b) and (b, a) belong to $\text{SI}_{\{p, q\}}$, then the formula $\theta_{p, u, q}(x, y) \wedge \theta_{q, v, p}(y, x)$ witnesses that $[a, b]$ is a (p, φ, q) -edge. If the edge $[a, b]$ is principal and $\theta_{p, u, q}(a, y)$ is an isolating formula such that $\models \theta_{p, u, q}(a, b)$, $\models p(a)$, then the label u is *invertible* and a

label $v \in U^{\geq 0}$ corresponds uniquely to u such that $\theta_{q,v,p}(b, y)$ is an isolating formula with $\models \theta_{q,v,p}(b, a)$, and vice versa. The labels u and v are *reciprocally inverse* and are denoted by v^{-1} and u^{-1} respectively. In general case, each label $u \in U^{\geq 0}$ has a (nonempty) set of *inverse* labels in $U^{\geq 0}$, denoted also by u^{-1} . Note that independently on a label $u \in U^{\geq 0}$, for which a formula $\theta_{p,u,q}(x, y)$ witnesses that $[a, b]$ is a (p, φ, q) -edge, the set u^{-1} includes all labels $v \in U^{\geq 0}$ such that $[b, a]$ is a $(q, \theta_{q,v,p}, p)$ -edge.

Neutral labels correspond, for instance, the formulas $\theta_{p,u,q}(x, y) \vee \theta_{p,v,q}(x, y)$, where $u < 0$ and $v \geq 0$.

For types $p_1, p_2, \dots, p_{k+1} \in S^1(\emptyset)$ and sets $X_1, X_2, \dots, X_k \subseteq U$ of labels we denote by

$$\text{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$$

the set of all labels $u \in U$ correspondent to formulas $\theta_{p_1, u, p_{k+1}}(x, y)$ satisfying, for realizations a of p_1 and some $u_1 \in X_1, \dots, u_k \in X_k$, the following condition:

$$\theta_{p_1, u, p_{k+1}}(a, y) \vdash \theta_{p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}}(a, y),$$

where

$$\begin{aligned} & \theta_{p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}}(x, y) \rightleftharpoons \\ & \rightleftharpoons \exists x_2, x_3, \dots, x_{k-1} (\theta_{p_1, u_1, p_2}(x, x_2) \wedge \theta_{p_2, u_2, p_3}(x_2, x_3) \wedge \dots \\ & \dots \wedge \theta_{p_{k-1}, u_{k-1}, p_k}(x_{k-1}, x_k) \wedge \theta_{p_k, u_k, p_{k+1}}(x_k, y)). \end{aligned}$$

In view of transitivity of semi-isolation, each formula $\theta_{p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}}(x, y)$ has a label in $\rho_{\nu(p_1, p_{k+1})}$.

Thus the Boolean $\mathcal{P}(U)$ of U is the universe of an *algebra* \mathfrak{A} of *distributions of binary semi-isolating formulas* with k -ary operations

$$\text{SI}(p_1, \cdot, p_2, \cdot, \dots, p_k, \cdot, p_{k+1}),$$

where $p_1, \dots, p_{k+1} \in S^1(\emptyset)$. This algebra has a natural restriction to any family $R \subseteq S^1(\emptyset)$ as well as to the algebras of distributions of binary *isolating* formulas [2]. Besides, if U_0 is a subalphabet of U then the restriction of the universe of \mathfrak{A} to the set $\mathcal{P}(U_0)$ and the restrictions for values of operations to the set U_0 forms, possibly partial, algebra $\mathfrak{A} \upharpoonright U_0$.

Note that if some set X_i is disjoint with $\rho_{\nu(p_i, p_{i+1})}$, in particular, if it is empty then

$$\text{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) = \emptyset,$$

and if each X_i has common elements with $\rho_{\nu(p_i, p_{i+1})}$ then

$$\text{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) \neq \emptyset.$$

Note also that if $X_i \not\subseteq \rho_{\nu(p_i, p_{i+1})}$ for some i then

$$\begin{aligned} & \text{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) = \\ & = \text{SI}(p_1, X_1 \cap \rho_{\nu(p_1, p_2)}, p_2, X_2 \cap \rho_{\nu(p_2, p_3)}, \dots, p_k, X_k \cap \rho_{\nu(p_k, p_{k+1})}, p_{k+1}). \end{aligned}$$

In view of the previous equation, further, considering values

$$\text{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}),$$

we shall assume that $X_i \subseteq \rho_{\nu(p_i, p_{i+1})}$, $i = 1, \dots, k$.

If each set X_i is a singleton consisting of an element u_i then we use u_i instead of X_i in $\text{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$ and write

$$\text{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}).$$

By the definition the following equality holds:

$$\begin{aligned} & \text{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) = \\ & = \cup \{ \text{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}) \mid u_1 \in X_1, \dots, u_k \in X_k \}. \end{aligned}$$

Hence the specification of $\text{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$ is reduced to the specifications of $\text{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1})$. Note also that $\text{SI}(p, X, q) = X$ for any $X \subseteq \rho_{\nu(p, q)}$.

Clearly, if $u_i = 0$ then $p_i = p_{i+1}$ for nonempty sets

$$\text{SI}(p_1, u_1, p_2, u_2, \dots, p_i, 0, p_{i+1}, \dots, p_k, u_k, p_{k+1})$$

and the following conditions hold:

$$\text{SI}(p_1, 0, p_1) = \{0\},$$

$$\begin{aligned} & \text{SI}(p_1, u_1, p_2, u_2, \dots, p_i, 0, p_{i+1}, \dots, p_k, u_k, p_{k+1}) = \\ & = \text{SI}(p_1, u_1, p_2, u_2, \dots, p_i, u_{i+1}, p_{i+2}, \dots, p_k, u_k, p_{k+1}). \end{aligned}$$

If all types p_i equal to a type p then we write $\text{SI}_p(X_1, X_2, \dots, X_k)$ and $\text{SI}_p(u_1, u_2, \dots, u_k)$ as well as $\lceil X_1, X_2, \dots, X_k \rceil_p$ and $\lceil u_1, u_2, \dots, u_k \rceil_p$ instead of

$$\text{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$$

and

$$\text{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1})$$

respectively. We omit the index \cdot_p if the type p is fixed. In this case, we write $\theta_{u_1, u_2, \dots, u_k}(x, y)$ instead of $\theta_{p, u_1, p, u_2, \dots, p, u_k, p}(x, y)$.

Proposition 1.1. (1) If $p, q \in S^1(T)$ are principal types then $\rho_{\nu(p, q)} \cup \rho_{\nu(q, p)} \subseteq U^{\geq 0}$.

(2) If $p, q \in S^1(T)$, p is a principal type and q is a non-principal type then $\rho_{\nu(p, q)} = \emptyset$ and $\rho_{\nu(q, p)} \subseteq U^-$.

Proof. (1) If $\rho_{\nu(p, q)}$ contains a label $u \notin U^{\geq 0}$ then there are realizations a and b of p and q respectively such that $(a, b) \in \text{SI}_{\{p, q\}}$ and $(b, a) \notin \text{SI}_{\{p, q\}}$. But since $p(x)$ contains a principal formula $\varphi(x)$, this formula witnesses that $(b, a) \in \text{SI}_{\{p, q\}}$. The contradiction implies that $\rho_{\nu(p, q)} \subseteq U^{\geq 0}$. Similarly we obtain $\rho_{\nu(q, p)} \subseteq U^{\geq 0}$.

(2) Let $\varphi(x)$ be a principal formula of $p(x)$. If $\models p(a)$, $\models q(b)$, and $(a, b) \in \text{SI}_{\{p, q\}}$ that witnessed by a formula $\theta_u(x, y)$, the formula $\exists x(\varphi(x) \wedge \theta_u(x, y))$ isolates $q(y)$. Since q is not isolated we obtain $\rho_{\nu(p, q)} = \emptyset$. By the same reason, $\rho_{\nu(q, p)} \subseteq U^-$. \square

Corollary 1.2. If $p(x)$ is a principal type then $\rho_{\nu(p)} \subseteq U^{\geq 0}$.

Proposition 1.3. Let p_1, p_2, \dots, p_{k+1} be types in $S^1(\emptyset)$. The following assertions hold.

(1) If $u_i \in \rho_{\nu(p_i, p_{i+1})}$, $i = 1, \dots, k$, and some u_i is negative then

$$\text{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}) \subseteq U^-.$$

(2) If $u_i \in \rho_{\nu(p_i, p_{i+1})} \cap U^{\geq 0}$, $i = 1, \dots, k$, then

$$\text{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}) \subseteq U^{\geq 0}.$$

(3) If $u_i \in \rho_{\nu(p_i, p_{i+1})} \cap (U^{\geq 0} \cup U')$, $i = 1, \dots, k$, and some u_i belongs to U' then

$$\text{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}) \subseteq U'.$$

(4) If $u_i \in \rho_{\nu(p_i, p_{i+1})} \cap U^{\geq 0}$, $i = 1, \dots, k$, then all elements of the set $X \Leftarrow \text{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1})$ are invertible and the set $X^{-1} \Leftarrow \cup \{v^{-1} \mid v \in X\}$ coincides with the set $\text{SI}(p_{k+1}, u_k^{-1}, p_k, u_{k-1}^{-1}, \dots, p_2, u_1^{-1}, p_1)$.

Proof. (1)–(3) follow by the transitivity of semi-isolation.

(4) All elements in X are invertible by (2). Let v' be an element in $v^{-1} \subseteq X^{-1}$. Then for any $(p_1, \theta_{p_1, v, p_{k+1}}, p_{k+1})$ -edge $[a, b]$ the following conditions hold:

(a) there are realizations a_i of p_i , $i = 1, \dots, k+1$, such that $a_0 = a$, $a_{k+1} = b$, $\models \theta_{p_i, u_i, p_{i+1}}(a_i, a_{i+1})$, $i = 1, \dots, k$;

(b) $[b, a]$ is a $(p_{k+1}, \theta_{p_{k+1}, v', p_1}, p_1)$ -edge.

Since $[a_{i+1}, a_i]$ is an u'_i -edge for any $u'_i \in u_i^{-1}$, $i = 1, \dots, k$, then

$$\theta_{p_{k+1}, v', p_1}(b, x) \vdash \theta_{p_{k+1}, u'_k, p_k, u'_{k-1}, \dots, p_2, u'_1, p_1}(b, x),$$

whence, $v' \in \text{SI}(p_{k+1}, u_k^{-1}, p_k, u_{k-1}^{-1}, \dots, p_2, u_1^{-1}, p_1)$.

If $v' \in \text{SI}(p_{k+1}, u'_k, p_k, u'_{k-1}, \dots, p_2, u_1^{-1}, p_1)$, $u'_i \in u_i^{-1}$, $i = 1, \dots, k$, then $v' \geq 0$ and for any $(p_{k+1}, \theta_{p_{k+1}, v', p_1}, p_1)$ -edge $[b, a]$ there are realizations b_i of p_i , $i = 1, \dots, k+1$, such that $b_{k+1} = b$, $b_1 = a$, $\models \theta_{p_{i+1}, u'_i, p_i}(b_{i+1}, b_i)$, $i = 1, \dots, k$. We have $\models \theta_{p_i, u_i, p_{i+1}}(b_i, b_{i+1})$, $i = 1, \dots, k$, and so the elements b_1, \dots, b_{k+1} witness that $[a, b]$ is a $(p_1, \theta_{p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}}, p_{k+1})$ -edge. Then v' belongs to v^{-1} , where $v \in X$ is a label for the formula $\theta_{p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}}$. Thus, $v' \in X^{-1}$. \square

Corollary 1.4. *Restrictions of U to the sets $U^{\leq 0}$, $U^{\geq 0}$, and $U^{\geq 0} \cup U'$ form subalgebras of the algebra of distributions of binary semi-isolating formulas. The operation of inversion is coordinated with the operations of the algebra.*

2. Preordered algebras of distributions of binary semi-isolating formulas

For the set U of labels in the algebra \mathfrak{A} of binary semi-isolating formulas of theory T , we define the following relation \trianglelefteq : if $u, v \in U$ then $u \trianglelefteq v$ if and only if $u = v$, or $u, v \in \rho_{\nu(p, q)}$ for some types $p, q \in S^1(\emptyset)$ and $\theta_{p, u, q}(a, y) \vdash \theta_{p, v, q}(a, y)$ for some (any) realization a of p . If $u \trianglelefteq v$ and $u \neq v$ we write $u \triangleleft v$.

By the definition the relation \trianglelefteq is reflexive and transitive. It is antisymmetric since distinct labels correspond to non-equivalent formulas.

Below we consider some properties for the substructures $\langle \rho_{\nu(p, q)}; \trianglelefteq \rangle$ of the partially ordered set $\langle U; \trianglelefteq \rangle$.

Proposition 2.1. (1) *For any types $p, q \in S^1(\emptyset)$ the partially ordered set $\langle \rho_{\nu(p, q)}; \trianglelefteq \rangle$ forms a upper semilattice.*

(2) *An element $u \in \rho_{\nu(p, q)}$ is \trianglelefteq -minimal if and only if for a realization a of p , the formula $\theta_{p, u, q}(a, y)$ is isolating.*

(3) (monotony). If $u, v \in \rho_{\nu(p,q)}$ and $u \trianglelefteq v$ then $v \in U^\delta$, $\delta \in \{-, +\}$, implies $u \in U^\delta$, and if $u \in U'$ then $v \in U'$.

Proof. (1) If $u_1, u_2 \in \rho_{\nu(p,q)}$ then for the formulas $\theta_{p,u_1,q}(x, y)$ and $\theta_{p,u_2,q}(x, y)$ the label v for the formula $\theta_{p,u_1,q}(x, y) \vee \theta_{p,u_2,q}(x, y)$ is a supremum for the labels u_1 and u_2 .

(2) If $\theta_{p,u,q}(a, y)$ is an isolating formula then the label u is \trianglelefteq -minimal by the definition. If the formula $\theta_{p,u,q}(a, y)$ is not isolating then there is a formula $\varphi(a, y)$ such that the semi-isolating formulas $\theta_{p,u,q}(a, y) \wedge \varphi(a, y)$ and $\theta_{p,u,q}(a, y) \wedge \neg\varphi(a, y)$ are consistent. For the labels v_1 and v_2 of these formulas, we have $v_1 \neq v_2$, $v_1 \triangleleft u$, and $v_2 \triangleleft u$.

(3) If $v \in \rho_{\nu(p,q)} \cap U^-$ then for any solution b of the formula $\theta_{p,v,q}(a, y)$, where $\models p(a)$, the pair (a, b) is an irreversible arc. Hence, for any solution b of $\theta_{p,u,q}(a, y)$, where $u \trianglelefteq v$, the pair (a, b) is also an irreversible arc and so u belongs to U^- . Replacing arcs by edges, the same arguments show that $u \trianglelefteq v$ and $v \in U^+$ imply $u \in U^+$. If $u \in U'$ then the set of pairs (a, b) for the formula $\theta_{p,u,q}(a, y)$ contains both irreversible and reversible arcs. This property is preserved for any label v with $u \trianglelefteq v$, whence $v \in U'$. \square

The partial order \trianglelefteq has a natural extension to a preorder on the set $\mathcal{P}(U)$: for any sets $X, Y \in \mathcal{P}(U)$ we put $X \trianglelefteq Y$ if $X = \emptyset$, or for any $x \in X$ there is $y \in Y$ with $x \trianglelefteq y$ and for any $y \in Y$ there is $x \in X$ with $x \trianglelefteq y$. Thus, the algebra \mathfrak{A} is transformed to the preordered algebra $\langle \mathfrak{A}; \trianglelefteq \rangle$ with the monotonic property with respect its restrictions to the sets $U^{\leq 0}$, $U^{\geq 0}$, and U' .

Another natural expansion of the now preordered algebra $\langle \mathfrak{A}; \trianglelefteq \rangle$ is based on the the properties mentioned that if $u_1, u_2 \in \rho_{\nu(p,q)}$ and $v \in \rho_{\nu(q,r)}$ then the formulas $\theta_{p,u_1,q,v,r}(x, y)$ and $\theta_{p,u_1,q}(x, y) \vee \theta_{p,u_2,q}(x, y)$ as well as $\theta_{p,u_1,q}(x, y) \wedge \theta_{p,u_2,q}(x, y)$ and $\theta_{p,u_1,q}(x, y) \wedge \neg\theta_{p,u_2,q}(x, y)$ (if the formulas $\theta_{p,u_1,q}(a, y) \wedge \theta_{p,u_2,q}(a, y)$ and $\theta_{p,u_1,q}(a, y) \wedge \neg\theta_{p,u_2,q}(a, y)$ are consistent for $\models p(a)$) have labels in U . We denote these labels by $u_1 \circ v$, $u_1 \vee u_2$, $u_1 \wedge u_2$, and $u_1 \wedge \neg u_2$ respectively. The last label is also denoted by $\neg u_2 \wedge u_1$. The label $u_1 \circ v$ is the *composition* of labels u_1 and v ; $u_1 \vee u_2$ is the *union* or the *disjunction* of labels u_1 and u_2 ; $u_1 \wedge u_2$ is their *intersection* or *conjunction*; $u_1 \wedge \neg u_2$ is the *relative complement* of u_2 in u_1 .

Clearly, $u_1 \trianglelefteq u_1 \vee u_2$, $u_2 \trianglelefteq u_1 \vee u_2$, $u_1 \wedge u_2 \trianglelefteq u_1$, $u_1 \wedge u_2 \trianglelefteq u_2$, $u_1 \wedge \neg u_2 \trianglelefteq u_1$.

We set

$$(p, (u_1 \circ v), r) = \begin{cases} \{u_1 \circ v\}, & \text{if } u_1 \in \rho_{\nu(p,q)} \text{ and } v \in \rho_{\nu(q,r)}, \\ \emptyset, & \text{if } u_1 \notin \rho_{\nu(p,q)} \text{ or } v \notin \rho_{\nu(q,r)}, \end{cases}$$

$$\begin{aligned}
(p, (u_1 \vee u_2), q) &= \begin{cases} \{u_1 \vee u_2\}, & \text{if } u_1 \in \rho_{\nu(p,q)} \text{ and } u_2 \in \rho_{\nu(p,q)}, \\ \{u_1\}, & \text{if } u_1 \in \rho_{\nu(p,q)} \text{ and } u_2 \notin \rho_{\nu(p,q)}, \\ \{u_2\}, & \text{if } u_1 \notin \rho_{\nu(p,q)} \text{ and } u_2 \in \rho_{\nu(p,q)}, \\ \emptyset, & \text{if } u_1 \notin \rho_{\nu(p,q)} \text{ and } u_2 \notin \rho_{\nu(p,q)}, \end{cases} \\
(p, (u_1 \wedge u_2), q) &= \begin{cases} \{u_1 \wedge u_2\}, & \text{if } u_1 \in \rho_{\nu(p,q)}, u_2 \in \rho_{\nu(p,q)} \\ & \text{and } \models \exists y(\theta_{p,u_1,q}(a, y) \wedge \theta_{p,u_2,q}(a, y)), \\ \emptyset, & \text{otherwise,} \end{cases} \\
(p, (u_1 \wedge \neg u_2), q) &= \begin{cases} \{u_1 \wedge \neg u_2\}, & \text{if } u_1 \in \rho_{\nu(p,q)}, u_2 \in \rho_{\nu(p,q)} \\ & \text{and } \models \exists y(\theta_{p,u_1,q}(a, y) \wedge \neg \theta_{p,u_2,q}(a, y)), \\ \emptyset, & \text{otherwise,} \end{cases} \\
(p, (X_1 \tau X_2), q) &= \cup \{(p, (u_1 \tau u_2), q) \mid u_1 \in X_1, u_2 \in X_2\}, \tau \in \{\circ, \vee, \wedge\}, \\
(p, (X_1 \wedge \neg X_2), q) &= (p, (\neg X_2 \wedge X_1), q) = \\
&= \cup \{(p, (u_1 \wedge \neg u_2), q) \mid u_1 \in X_1, u_2 \in X_2\}, X_1, X_2 \in \mathcal{P}(U).
\end{aligned}$$

Labels u_1 and u_2 are *consistent* if $u_1 \wedge u_2 \in U$. If $u_1 \wedge u_2 = \emptyset$ the labels u_1 and u_2 are called *inconsistent*.

The preordered algebra $\langle \mathfrak{A}; \trianglelefteq \rangle$ equipped with binary operations $(p, (\cdot \tau \cdot), q)$, $\tau \in \{\vee, \wedge, \circ\}$, and $(p, (\cdot \wedge \neg \cdot), q)$, $p, q \in S^1(\emptyset)$, is called a *preordered algebra with relative set-theoretic operations and the composition* or briefly a *POSTC-algebra*.

For any types $p, q \in S^1(\emptyset)$ the structure $\langle \rho_{\nu(p,q)} \cup \{\emptyset\}; \vee, \wedge, \emptyset \rangle$ with operations \vee and \wedge on labels, being extended by equalities $u \vee \emptyset = u$, $u \wedge \emptyset = \emptyset$, where $u \in \rho_{\nu(p,q)} \cup \{\emptyset\}$, is an Ershov algebra, i. e., a *distributive lattice with zero \emptyset and relative complements* [17] such that for any $u, v \in \rho_{\nu(p,q)}$ if $u \trianglelefteq v$ and $u' = \neg u \wedge v$ is a label then $u \wedge u' = \emptyset$ and $u \vee u' = v$, and if the label u' does not exist then $u = v$.

A label $u \in U$ is an *atom* or an *atomic label* if u is a \trianglelefteq -minimal element in U , i. e., for any label $v \in U$ if $v \trianglelefteq u$ then $v = u$.

By Proposition 2.1, the set of atoms equals the set of isolating labels and, thus, each atom $u \in \rho_{\nu(p,q)}$ is represented by an isolated formula $\theta_{p,u,q}(a, y)$, where $\models p(a)$.

Let R be a nonempty family of types in $S^1(\emptyset)$, \mathfrak{A}_R be a restriction of POSTC-algebra \mathfrak{A} to the family R . The structure \mathfrak{A}_R is *atomic* if for any types $p, q \in R$ and for any label $u \in \rho_{\nu(p,q)}$ there is an atom $v \in \rho_{\nu(p,q)}$ such that $v \trianglelefteq u$. The POSTC-algebra \mathfrak{A} is called *R-atomic* if \mathfrak{A}_R is atomic. If $R = S^1(\emptyset)$ then the *R-atomic* POSTC-algebra is called *atomic*.

Using the definition of atomic structure, of R -atomic POSTC-algebra, and of small theory we obtain the following assertions.

Proposition 2.2. *If R is a nonempty family of types in $S^1(\emptyset)$ and for any type $p \in R$, there is an atomic model \mathcal{M}_p over a realization of p , then the POSTC-algebra \mathfrak{A} is R -atomic.*

Corollary 2.3. *If T is a small theory then the POSTC-algebra \mathfrak{A} is atomic.*

3. Ranks and degrees of semi-isolation

The following definition is a local variation of Morley rank [18].

Definition. For triples (p, u, q) , where $p, q \in S^1(\emptyset)$, $u \in U \cup \{\emptyset\}$, we define inductively the *rank* $\text{si}(p, u, q)$ of *semi-isolation*:

- (1) $\text{si}(p, u, q) = 0$ if $u \notin \rho_{\nu(p, q)}$;
- (2) $\text{si}(p, u, q) \geq 1$ if $u \in \rho_{\nu(p, q)}$;
- (3) for a positive ordinal α , $\text{si}(p, u, q) \geq \alpha + 1$ if there is a set $\{v_i \mid i \in \omega\}$ of pairwise inconsistent labels such that $v_i \triangleleft u$ and $\text{si}(p, v_i, q) \geq \alpha$, $i \in \omega$;
- (4) for a limit ordinal α , $\text{si}(p, u, q) \geq \alpha$ if $\text{si}(p, u, q) \geq \beta$ for any $\beta \in \alpha$.

As usual, we write $\text{si}(p, u, q) = \alpha$ if $\text{si}(p, u, q) \geq \alpha$ and $\text{si}(p, u, q) \not\geq \beta$ for $\alpha \in \beta$; $\text{si}(p, u, q) = \infty$ if $\text{si}(p, u, q) \geq \alpha$ for any ordinal α .

If types p and q are fixed, we write $\text{si}(u)$ instead of $\text{si}(p, u, q)$ and this value is said to be the *rank of semi-isolation* or the *si-rank* of the label u or of the element $u = \emptyset$ (with respect to the pair (p, q)). For a formula $\theta_{p, u, q}(x, y)$ we set $\text{si}(\theta_{p, u, q}(x, y)) = \text{si}(u)$.

Clearly, if the theory is small then the si-rank of any label is an ordinal (having a label u with $\text{si}(p, u, q) = \infty$, we get continuum many complete types $r(x, y) \supset p(x) \cup q(y)$).

By the definition we have the following inequality for any formula $\theta_{p, u, q}(x, y)$ and any realization a of p giving a low bound for Morley rank of the formula $\theta_{p, u, q}(a, y)$ by the si-rank:

$$\text{si}(\theta_{p, u, q}(x, y)) \leq \text{MR}(\theta_{p, u, q}(a, y)) + 1. \quad (1)$$

The inequality (1) implies

Remark 3.1. If a theory T has a finite Morley rank then si-ranks of labels $\bigcup_{p, q \in S^1(T)} \rho_{\nu(p, q)}$ are bounded by the value $\text{MR}(x \approx x) + 1$.

We set $\text{si}(p, q) \doteq \sup\{\text{si}(p, u, q) \mid u \in U \cup \{\emptyset\}\}$, $\text{si}(p) \doteq \text{si}(p, p)$. For a nonempty family R of 1-types, we put $\text{si}(R) \doteq \sup\{\text{si}(p, q) \mid p, q \in R\}$. A family R is called *si-minimal* if $\text{si}(R) = 1$. The value $\text{si}(p, q)$ is said to be the *rank of semi-isolation* or the *si-rank* of pair (p, q) , and $\text{si}(R)$ is the *rank of semi-isolation* or the *si-rank* of the family R .

Since there are $|T|$ formulas of a theory T and the inequality (1) holds we obtain

Proposition 3.2. *Each si-rank in a theory T is either equal to ∞ or less than $\min\{|T|^+, (\text{MR}(x \approx x) + 1)^+\}$. If Morley rank $\text{MR}(x \approx x)$ is equal to an ordinal α then any si-rank in T is not more than $\alpha + 1$.*

The estimation for si-ranks in Proposition 3.2 can be far from exact. For instance, si-ranks in ω -categorical theories are finite while there are non- ω -stable ω -categorical theories.

Proposition 3.3. *For any types $p, q \in S^1(\emptyset)$ the following assertions are satisfied.*

- (1) *If $u, v \in \rho_{\nu(p, q)} \cup \{\emptyset\}$ and $u \leq v$ then $\text{si}(u) \leq \text{si}(v)$.*
- (2) *If $u, v \in \rho_{\nu(p, q)} \cup \{\emptyset\}$ then $\text{si}(u \vee v) = \max\{\text{si}(u), \text{si}(v)\}$ and $\text{si}(u \wedge v) \leq \min\{\text{si}(u), \text{si}(v)\}$. The last inequality is transformed to the equality if and only if there is a label v' such that $v' \leq u$, $v' \leq v$, and $\text{si}(v') = \text{si}(u)$ or $\text{si}(v') = \text{si}(v)$.*
- (3) *The equality $\text{si}(p, q) = 0$ holds if and only if there are no realizations of p semi-isolating realizations of q .*
- (4) *The equality $\text{si}(p, q) = 1$ holds if and only if there is a $(p \rightarrow q)$ -formula and each $(p \rightarrow q)$ -formula $\varphi(x, y)$ is equivalent to a disjunction of formulas $\varphi_i(x, y)$ such that each formula $\varphi_i(a, y)$ is isolating, where $\models p(a)$.*

Proof is obvious. \square

Proposition 3.4. *For any nonempty family $R \subseteq S^1(\emptyset)$ the following assertions are satisfied.*

- (1) $\text{si}(R) \geq 1$.
- (2) *The family R is si-minimal if and only if for any types $p, q \in R$ each $(p \rightarrow q)$ -formula $\varphi(x, y)$ is equivalent to a disjunction of formulas $\varphi_i(x, y)$ such that each formula $\varphi_i(a, y)$ is isolating, where $\models p(a)$.*

Proof. (1) is implied by the inequality $\text{si}(p) \geq 1$ for any type $p \in S^1(\emptyset)$ since the formula $(a \approx y)$ witnesses that a semi-isolates itself, where $\models p(a)$. (2) is an obvious corollary of (1) and Proposition 3.3, (4). \square

Remark 3.5. Since for a strongly minimal theory T the set of solutions for any formula $\varphi(a, y)$ is finite or cofinite, any semi-isolating formula $\psi(a, y)$ is represented as a finite disjunction of some isolating formulas $\psi_i(a, y)$ or as a negation of a finite disjunction of isolating formulas $\psi_i(a, y)$. If $\psi(a, y) \vdash p(y)$ and $p(y)$ is a non-principal type then the representation of $\psi(a, y)$ is possible only as a finite disjunction of isolating formulas. It means that $\text{si}(p) = 1$. If $p(y)$ is a principal type and there are finitely many pairwise non-equivalent isolating formulas $\psi(a, y)$ with $\models p(a)$ and $\psi(a, y) \vdash p(y)$ then $\text{si}(p) = 1$ too. If there are infinitely many these pairwise non-equivalent isolating formulas $\psi(a, y)$ then $\text{si}(p) = 2$. \square

Definition. Let α be a positive ordinal, u_1 and u_2 be labels in $\rho_{\nu(p,q)}$ such that $\text{si}(u_1) = \text{si}(u_2) = \alpha$. The labels u_1 and u_2 are α -almost identic or \sim_α -equivalent (denoted by $u_1 \sim_\alpha u_2$) if $\text{si}(u_1 \div u_2) < \alpha$, where $u_1 \div u_2 \equiv (u_1 \wedge \neg u_2) \vee (\neg u_1 \vee u_2)$.

Proposition 3.6. *The relation \sim_α is an equivalence relation for any set of labels in $\rho_{\nu(p,q)}$ having the si-rank α .*

Proof. Clearly the relation \sim_α is reflexive and symmetric. For the checking of transitivity we assume that $u_1 \sim_\alpha u_2$ and $u_2 \sim_\alpha u_3$. Since $(u_1 \wedge \neg u_2 \wedge u_3) \leq (u_1 \wedge \neg u_2) \leq (u_1 \div u_2)$ we have

$$\text{si}(u_1 \wedge \neg u_2 \wedge u_3) \leq \text{si}(u_1 \div u_2) < \alpha.$$

As $u_1 \wedge u_3 = (u_1 \wedge u_2 \wedge u_3) \vee (u_1 \wedge \neg u_2 \wedge u_3)$ and $\text{si}(u_1 \wedge \neg u_2 \wedge u_3) < \alpha$, for $u_1 \sim_\alpha u_3$, it is enough to prove that $\text{si}(u_1 \wedge u_2 \wedge u_3) = \alpha$. Suppose on contrary that $\text{si}(u_1 \wedge u_2 \wedge u_3) < \alpha$. Then $\text{si}(u_1 \wedge u_2) = \alpha$ and

$$u_1 \wedge u_2 = (u_1 \wedge u_2 \wedge u_3) \vee (u_1 \wedge u_2 \wedge \neg u_3)$$

imply $\text{si}(u_1 \wedge u_2 \wedge \neg u_3) = \alpha$. But $(u_1 \wedge u_2 \wedge \neg u_3) \leq (u_2 \wedge \neg u_3) \leq (u_2 \div u_3)$, and $\text{si}(u_2 \div u_3) < \alpha$ gives $\text{si}(u_1 \wedge u_2 \wedge \neg u_3) < \alpha$. The obtained contradiction means that $u_1 \sim_\alpha u_3$. \square

By the definition, for any label $u \in \rho_{\nu(p,q)}$ having the si-rank α , there is a greatest number $n \in \omega \setminus \{0\}$ of pairwise inconsistent (or, that equivalent, of pairwise non- \sim_α -equivalent) labels u_1, \dots, u_n such that $u_i \leq u$ and $\text{si}(u_i) = \alpha$, $i = 1, \dots, n$. This number n is called the *degree of semi-isolation* or the *si-degree* of label u and it is denoted by $\deg(p, u, q)$ or by $\deg(u)$. We have $\text{si}(\emptyset) = 0$ and put $\deg(\emptyset) = 1$.

Proposition 3.7. (1) If $u \in \rho_{\nu(p,q)}$ and $\text{si}(u) = \alpha$ then $\deg(u)$ is equal to the number of pairwise inconsistent labels $u_1, \dots, u_n \in \rho_{\nu(p,q)}$ having the si-rank α , the si-degree 1, and such that $u = u_1 \vee \dots \vee u_n$.
(2) If $u, v \in \rho_{\nu(p,q)}$, $\text{si}(u) = \text{si}(v)$, and $u \leq v$ then $\deg(u) \leq \deg(v)$.
(3) If $u, v \in \rho_{\nu(p,q)}$ and $\text{si}(u) = \text{si}(v)$ then

$$\deg(u \vee v) \leq \deg(u) + \deg(v).$$

The equality in this inequality holds if and only if $\text{si}(u \wedge v) < \text{si}(u)$. If $\text{si}(u \wedge v) = \text{si}(u)$ then

$$\deg(u \vee v) = \deg(u) + \deg(v) - \deg(u \wedge v).$$

(4) If $u \in \rho_{\nu(p,q)}$ is a label for an isolating formula, i. e., u is an atom, then $\text{si}(u) = 1$ and $\deg(u) = 1$.
(5) If for a label $u \in \rho_{\nu(p,q)}$, $\text{si}(u) = 1$ and $\deg(u) = 1$, then u is not neutral.
(6). If $u \in \rho_{\nu(p,q)}$ and $\text{si}(u) = 1$ then $\deg(u)$ is equal to the number of pairwise inconsistent labels $u_1, \dots, u_n \in \rho_{\nu(p,q)}$ for isolating formulas such that $u = u_1 \vee \dots \vee u_n$.

Proof is obvious. \square

If there is a label $u \in \rho_{\nu(p,q)}$ with $\text{si}(p, q) = \text{si}(u)$ then the *degree of semi-isolation* or the *si-degree* $\deg(p, q)$ of pair (p, q) is

$$\sup\{\deg(u) \mid u \in \rho_{\nu(p,q)}, \text{si}(p, q) = \text{si}(u)\},$$

$$\deg(p) \equiv \deg(p, p).$$

If for a nonempty family R of 1-types there is a label $u \in \rho_{\nu(p,q)}$, $p, q \in R$, with $\text{si}(R) = \text{si}(u)$ then the *degree of semi-isolation* or the *si-degree* $\deg(R)$ of R is

$$\sup\{\deg(u) \mid u \in \rho_{\nu(R)}, \text{si}(R) = \text{si}(u)\}.$$

Clearly, if $\deg(p, q)$ or $\deg(R)$ exist then these values are positive natural numbers or equal ω .

For an ordinal α , a natural number $n \geq 1$, and a set $X \in \{U, U \cup \{\emptyset\}\}$ we put

$$X \upharpoonright (\alpha, n) \equiv \{u \in X \mid \text{si}(u) \leq \alpha \text{ and if } \text{si}(u) = \alpha \text{ then } \deg(u) < n\},$$

$$X \upharpoonright (\alpha, \omega) \rightleftharpoons X \upharpoonright \alpha \rightleftharpoons \bigcup_{n \geq 1} X \upharpoonright (\alpha, n).$$

Clearly, if $\alpha = \beta + 1$ then $X \upharpoonright (\alpha, 1) = X \upharpoonright \beta$, and if α is a limit ordinal then $X \upharpoonright (\alpha, 1) = \bigcup_{\beta < \alpha} X \upharpoonright \beta$.

For ordinals α, β , $\beta \in (\omega + 1) \setminus \{0\}$, and for the algebra \mathfrak{A} of distributions of binary semi-isolating formulas of theory T as well as for expansions and restrictions \mathfrak{A}' of \mathfrak{A} , defined in previous sections, we denote by $\mathfrak{A} \upharpoonright (\alpha, \beta)$ and $\mathfrak{A}' \upharpoonright (\alpha, \beta)$ as well as by $\mathfrak{A}_{\alpha, \beta}$ and $\mathfrak{A}'_{\alpha, \beta}$ the restrictions of these algebras to the set $(U \cup \{\emptyset\}) \upharpoonright (\alpha, \beta)$. If $\beta = \omega$, these restrictions are denoted by $\mathfrak{A} \upharpoonright \alpha$, $\mathfrak{A}' \upharpoonright \alpha$, \mathfrak{A}_α , and \mathfrak{A}'_α . The restrictions are called the (α, β) -restrictions and the α -restrictions respectively.

Since the si-rank of each label is positive, non-trivial restrictions (i. e., with nonempty sets of used labels) are only the restrictions of algebras with $\alpha > 0$. If $\text{si}(S^1(\emptyset)) = \alpha_0$ then, taking into consideration the inequality $\alpha > 0$, all essential (i. e., reflecting links of sets of labels of semi-isolating formulas with respect to their si-ranks) restrictions of these algebras are formed only for $0 < \alpha < \alpha_0$.

In view of Proposition 3.7 we obtain

Proposition 3.8. *The algebra of distributions of binary isolating formulas of theory T coincides with the algebra $\mathfrak{A} \upharpoonright (1, 2)$. The algebra $\mathfrak{A} \upharpoonright 1$ consists of labels being disjunctions of labels of isolating formulas.*

4. Monoid of distributions of binary semi-isolating formulas on a set of realizations of a type

Consider a complete theory T , a type $p(x) \in S(T)$, a regular labelling function $\nu(p): \text{PF}(p)/\text{PE}(p) \rightarrow U$, and a family of sets $\text{SI}_p(u_1, \dots, u_k)$ of labels of binary semi-isolating formulas, $u_1, \dots, u_k \in \rho_{\nu(p)}$, $k \in \omega$.

Below we show some basic properties for sets

$$[u_1, \dots, u_k] \rightleftharpoons \text{SI}_p(u_1, \dots, u_k).$$

Proposition 4.1 (Associativity). *For any $u_1, u_2, u_3 \in \rho_{\nu(p)}$, the following equalities hold:*

$$[[u_1, u_2], u_3] = [u_1, u_2, u_3] = [u_1, [u_2, u_3]].$$

Proof of inclusions $\lceil [u_1, u_2], u_3 \rceil \subseteq \lceil u_1, u_2, u_3 \rceil$ and $\lceil u_1, \lceil u_2, u_3 \rceil \rceil \subseteq \lceil u_1, u_2, u_3 \rceil$ repeats [2, Proposition 3.1, 4].

The reverse inclusions are satisfied since, taking labels v_1 and v_2 for the formulas $\theta_{u_1, u_2}(x, y)$ and $\theta_{u_2, u_3}(x, y)$, we obtain, for $\models p(a)$, that the formulas $\theta_{v_1, u_3}(a, y)$, $\theta_{u_1, u_2, u_3}(a, y)$, and $\theta_{u_1, v_2}(a, y)$ are pairwise equivalent, i. e.,

$$\lceil v_1, u_3 \rceil = \lceil u_1, u_2, u_3 \rceil = \lceil u_1, v_2 \rceil. \quad \square$$

In view of associativity, using the induction on number of parenthesis, we prove that all operations $\lceil \cdot, \cdot, \dots, \cdot \rceil$ acting on sets in $\mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}$ are generated by the binary operation $\lceil \cdot, \cdot \rceil$ on the set $\mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}$ and the values $\lceil X_1, X_2, \dots, X_k \rceil$, $X_1, X_2, \dots, X_k \subseteq \rho_{\nu(p)}$, do not depend on the sequence of adding of brackets for

$$X_{i, i+1, \dots, i+m+n} \rightleftharpoons \lceil X_{i, i+1, \dots, i+m}, X_{i+m+1, i+m+2, \dots, i+m+n} \rceil,$$

where $X_{1, 2, \dots, k} = \lceil X_1, X_2, \dots, X_k \rceil$.

Thus the structure $\mathfrak{SI}_{\nu(p)} \rightleftharpoons \langle \mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}; \lceil \cdot, \cdot \rceil \rangle$ is a semigroup admitting the representation of all operations $\lceil \cdot, \cdot, \dots, \cdot \rceil$ by terms of the language $\lceil \cdot, \cdot \rceil$. Further the operation $\lceil \cdot, \cdot \rceil$ will be denoted also by \cdot and we shall use the record uv instead of $u \cdot v$.

Since by the choice of the label 0 for the formula $(x \approx y)$ the equalities $X \cdot \{0\} = X$ and $\{0\} \cdot X = X$ are true for any $X \subseteq \rho_{\nu(p)}$, the semigroup $\mathfrak{SI}_{\nu(p)}$ has the unit $\{0\}$, and it is a monoid. We have

$$Y \cdot Z = \bigcup \{yz \mid y \in Y, z \in Z\}$$

for any sets $Y, Z \in \mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}$ in this structure.

Thus the following proposition holds.

Proposition 4.2. *For any complete theory T , any type $p \in S(T)$, and the regular labelling function $\nu(p)$, any operation $\text{SI}_p(\cdot, \cdot, \dots, \cdot)$ on the set $\mathcal{P}(\rho_{\nu(p)}) \setminus \{\emptyset\}$ interpretable by a term of the monoid $\mathfrak{SI}_{\nu(p)}$.*

The monoid $\mathfrak{SI}_{\nu(p)}$ is called the *monoid of binary semi-isolating formulas over the labelling function $\nu(p)$* or the *$\text{SI}_{\nu(p)}$ -monoid*.

In view of Propositions 1.3 and 4.1 we obtain

Proposition 4.3. *For any complete theory T , any type $p \in S(T)$, and the regular labelling function $\nu(p)$, the restriction $\mathfrak{SI}_{\nu(p)}^{\leq 0}$ (respectively $\mathfrak{SI}_{\nu(p)}^{\geq 0}$,*

$\mathfrak{SI}_{\nu(p)}^{\geq 0, \text{neu}}$) of the monoid $\mathfrak{SI}_{\nu(p)}$ to the set $U^{\leq 0}$ ($U^{\geq 0}$, $U^{\geq 0} \cup U'$) is a submonoid of $\mathfrak{SI}_{\nu(p)}$.

By Proposition 3.8, the $(1, 2)$ -restriction of the monoid $\mathfrak{SI}_{\nu(p)}$ coincides with the $I_{\nu(p)}$ -groupoid $\mathfrak{P}_{\nu(p)}$. Besides, the $(1, 2)$ -restrictions of monoids $\mathfrak{SI}_{\nu(p)}^{\leq 0}$ and $\mathfrak{SI}_{\nu(p)}^{\geq 0}$ equal respectively to the groupoid $\mathfrak{P}_{\nu(p)}^{\leq 0}$ and the monoid $\mathfrak{P}_{\nu(p)}^{\geq 0}$.

5. α -deterministic and almost α -deterministic $\text{SI}_{\nu(p)}$ -monoids

In the following definition, we generalize the notions of deterministic and almost deterministic structure $\mathfrak{P}_{\nu(p)}$ proposed in [2].

Definition. Let U_0 be a subalphabet of the alphabet U , α be a positive ordinal, and $n \geq 1$ be a natural number. We put

$$\rho_{\nu(p), \alpha, n} \Rightarrow \{u \in \rho_{\nu(p)} \mid \text{si}(u) \leq \alpha, \deg(u) < n \text{ for } \text{si}(u) = \alpha\},$$

$$\rho_{\nu(p), \alpha} \Rightarrow \bigcup_{n \in \omega} \rho_{\nu(p), \alpha, n}.$$

The partial subalgebra $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ of the monoid $\mathfrak{SI}_{\nu(p)}$ is called (α, n) -deterministic if for any labels $u_1, u_2 \in \rho_{\nu(p), \alpha, n} \cap U_0$, the set $[u_1, u_2] \cap U_0$ consists of labels having the si-ranks $\leq \alpha$ and contains less than n pairwise non- \sim_α -equivalent labels of si-rank α .

The partial subalgebra $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ of the monoid $\mathfrak{SI}_{\nu(p)}$ is called α -deterministic if $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ is $(\alpha, 2)$ -deterministic.

The partial subalgebra $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ of the monoid $\mathfrak{SI}_{\nu(p)}$ is called *almost α -deterministic* or (α, ω) -deterministic if for any labels $u_1, u_2 \in \rho_{\nu(p), \alpha} \cap U_0$, the set $[u_1, u_2] \cap U_0$ consists of labels having the si-ranks $\leq \alpha$ and contains finitely many pairwise non- \sim_α -equivalent labels of si-rank α .

By the definition, each (α, ω) -deterministic structure $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ is a union of its (α, n) -deterministic substructures, $n \geq 1$. So each α -deterministic structure $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ is almost α -deterministic.

If $U_0 = U$ we shall not point out restrictions to the set U_0 for considered structures.

Below we show some basic properties of (almost) α -deterministic partial algebras $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$.

Proposition 5.1. (Monotony) *If a structure $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ is (almost) α -deterministic and β is a positive ordinal then the structure $(\mathfrak{SI}_{\nu(p)} \upharpoonright U_0) \upharpoonright \beta$ is also (almost) α -deterministic.*

Proof is obvious. \square

Proposition 5.2. *For any monoid $\mathfrak{SI}_{\nu(p)}$ and ordinals α, β , where $\alpha, \beta > 0$, $\beta \in \omega + 1$, the following conditions are equivalent:*

- (1) *the monoid $\mathfrak{SI}_{\nu(p)}$ is (α, β) -deterministic;*
- (2) *$\text{si}(u_1 \circ u_2) \leq \alpha$ for any labels $u_1, u_2 \in \rho_{\nu(p), \alpha, \beta}$ and if $\text{si}(u_1 \circ u_2) = \alpha$ then $\deg(u_1 \circ u_2) < \beta$.*

Proof. The implication (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (1). Consider arbitrary labels $u_1, u_2 \in \rho_{\nu(p), \alpha, \beta}$. Since, by hypothesis, $\text{si}(u_1 \circ u_2) \leq \alpha$ and $v \trianglelefteq (u_1 \circ u_2)$ for any label $v \in [u_1, u_2]$, $[u_1, u_2]$ consists of labels of si-ranks $\leq \alpha$, and if $\text{si}(v) = \text{si}(u_1 \circ u_2) = \alpha$ then $\deg(v) \leq \deg(u_1 \circ u_2) < \beta$. Thus, the monoid $\mathfrak{SI}_{\nu(p)}$ is (α, β) -deterministic. \square

Proposition 5.2 immediately implies

Corollary 5.3. *For any monoid $\mathfrak{SI}_{\nu(p)}$ and a positive ordinal α the following conditions are equivalent:*

- (1) *the monoid $\mathfrak{SI}_{\nu(p)}$ is almost α -deterministic;*
- (2) *$\text{si}(u_1 \circ u_2) \leq \alpha$ for any labels $u_1, u_2 \in \rho_{\nu(p), \alpha}$.*

Corollary 5.4. *If $\text{si}(p)$ is an ordinal then the monoid $\mathfrak{SI}_{\nu(p)}$ is almost $\text{si}(p)$ -deterministic.*

Proposition 5.5. *If a monoid $\mathfrak{SI}_{\nu(p)}$ is (α, β) -deterministic then the structure $\mathfrak{SI}_{\nu(p), \alpha} = \mathfrak{SI}_{\nu(p)} \upharpoonright \alpha$ is also an (α, β) -deterministic monoid.*

Proof. Since for any α -restriction the associativity, the presence of unit $\{0\}$, and the (α, β) -determinacy is preserved, it is enough to note that for any labels u_1 and u_2 in $\mathfrak{SI}_{\nu(p), \alpha, \beta}$ there is a label v in $\mathfrak{SI}_{\nu(p), \alpha, \beta}$ belonging $[u_1, u_2]$. We can take $u_1 \circ u_2$ for v since, by hypothesis, $\text{si}(u_1 \circ u_2) \leq \alpha$ and if $\text{si}(u_1 \circ u_2) = \alpha$ then $\deg(u_1 \circ u_2) < \beta$. \square

Proposition 5.6. *If $\text{si}(p)$ is an ordinal then the monoid $\mathfrak{SI}_{\nu(p)}$ is $\text{si}(p)$ -deterministic if and only if the value $\deg(p)$ is not defined or equals 1.*

Proof. If $\deg(p)$ is not defined the ordinal $\alpha = \text{si}(p)$ is limit and can not be achieved by labels in $\rho_{\nu(p)}$. In particular, for any $u_1, u_2 \in \rho_{\nu(p)}$ the set $[u_1, u_2]$ does not contain labels of si-rank α . If $\deg(p) \geq 2$ then there are

non- \sim_α -equivalent labels $u_1, u_2 \in \rho_{\nu(p)}$ of si-rank α . Then $[u_1 \vee u_2, 0]$ contains the labels u_1 and u_2 , whence the monoid $\mathfrak{SI}_{\nu(p)}$ is not α -deterministic. If $\deg(p) = 1$ then there is unique, up to \sim_α -equivalence, label in $\rho_{\nu(p)}$ having the si-rank α . Since such a label is unique, the monoid $\mathfrak{SI}_{\nu(p)}$ is α -deterministic. \square

Proposition 5.7. *The structure $\mathfrak{P}_{\nu(p)}$ is (almost) deterministic if and only if the structure $\mathfrak{SI}_{\nu(p),1,2}$ is (almost) 1-deterministic.*

Proof follows by the equality $\mathfrak{SI}_{\nu(p),1,2} = \mathfrak{P}_{\nu(p)}$. \square

Proposition 5.8. *Let $p(x)$ be a complete type of a theory T , $\nu(p)$ be a regular labelling function, and $\text{si}(p) < \omega$. The following conditions are equivalent:*

- (1) *the monoid $\mathfrak{SI}_{\nu(p)}$ is $(1, n)$ -deterministic for some $n \in \omega$;*
- (2) *the set $\rho_{\nu(p)}$ is finite;*
- (3) *the set $\rho_{\nu(p),1}$ is finite;*
- (4) *the set $\rho_{\nu(p),1,2}$ (consisting of all atoms $u \in \rho_{\nu(p)}$) is finite.*

Proof. If $\text{si}(p) > 1$ then, by $\text{si}(p) < \omega$, the set $\rho_{\nu(p),1}$ is infinite and so the set $\rho_{\nu(p)}$ is also infinite. Since each label in $\rho_{\nu(p),1}$ is a disjunction of labels in $\rho_{\nu(p),1,2}$ and for any labels $u_1, \dots, u_n \in \rho_{\nu(p),1}$ the label $u_1 \vee \dots \vee u_n$ belongs also to $\rho_{\nu(p),1}$, the set $\rho_{\nu(p),1,2}$ is infinite and the monoid $\mathfrak{SI}_{\nu(p)}$ is not $(1, n)$ -deterministic for $n \in \omega$. Thus, none of the conditions (1)–(4) is not satisfied.

If $\text{si}(p) = 1$ then each label in $\rho_{\nu(p)}$ has the si-rank 1 and is represented as a disjunction of labels in $\rho_{\nu(p),1,2}$. Thus, the conditions (2)–(4) are equivalent. If the set $\rho_{\nu(p),1,2}$ contains $m \in \omega$ labels then there are $2^m - 1$ labels forming the set $\rho_{\nu(p)}$. Hence, the monoid $\mathfrak{SI}_{\nu(p)}$ is $(1, 2^m - 1)$ -deterministic. If the set $\rho_{\nu(p),1,2}$ is infinite then, for pairwise distinct labels $u_1, \dots, u_m \in \rho_{\nu(p),1,2}$, the set $[u_1 \vee \dots \vee u_m, 0]$ contains $2^m - 1$ labels and, since m is not bounded, the monoid $\mathfrak{SI}_{\nu(p)}$ is not $(1, n)$ -deterministic for any n . Thus, the condition (1) is equivalent to each of the conditions (2)–(4). \square

Proposition 5.8 and [2, Corollary 7.4] imply

Corollary 5.9. *Let $p(x)$ be a complete type of a theory T , $\nu(p)$ be a regular labelling function, $\text{si}(p) < \omega$, and $\mathfrak{SI}_{\nu(p)}$ is a $(1, n)$ -deterministic monoid, for some $n \in \omega$, having a negative label. Then the groupoid $\mathfrak{SI}_{\nu(p),1,2}$ generates the strict order property.*

Definition [1, 19]. Let $p(x)$ be a type in $S(T)$. A type $q(x_1, \dots, x_n) \in S(T)$ is called a (n, p) -type if $q(x_1, \dots, x_n) \supseteq \bigcup_{i=1}^n p(x_i)$. The set of all (n, p) -types of T is denoted by $S_{n,p}(T)$ and elements of the set $S_p(T) \doteq \bigcup_{n \in \omega \setminus \{0\}} S_{n,p}(T)$ are called p -types.

A type $q(\bar{y})$ in $S_p(T)$ is called p -principal if there is a formula $\varphi(\bar{y}) \in q(\bar{y})$ such that $\cup\{p(y_i) \mid y_i \in \bar{y}\} \cup \{\varphi(\bar{y})\} \vdash q(\bar{y})$.

Lemma 5.10 [1, 19]. *For any type p and a natural number $n \geq 1$ the following conditions are equivalent:*

- (1) *the set of (n, p) -types with a tuple (x_1, \dots, x_n) of free variables is infinite;*
- (2) *there is a non- p -principal (n, p) -type.*

Proposition 5.8 and Lemma 5.10 imply

Corollary 5.11. *If $p(x)$ is a complete type of a theory T , $\nu(p)$ is a regular labelling function, and all $(2, p)$ -types are p -principal, then the monoid $\mathfrak{SI}_{\nu(p)}$ is $(1, n)$ -deterministic for some $n \in \omega$.*

By Corollaries 5.9 and 5.11, we obtain

Corollary 5.12. *Let $p(x)$ be a complete type of a theory T , $\nu(p)$ be a regular labelling function, $\rho_{\nu(p)} \cap U^- \neq \emptyset$, and all $(2, p)$ -types be p -principal. Then the groupoid $\mathfrak{SI}_{\nu(p), 1, 2}$ generates the strict order property.*

For a type $p(x)$ and a positive ordinal α , we denote by $\text{SI}_{p, \alpha}$ (in a model \mathcal{M} of T) the relation of semi-isolation (over \emptyset) on a set of realizations of p restricted to the set of formulas of si-rank $\leq \alpha$:

$$\text{SI}_{p, \alpha} \doteq \{(a, b) \mid \mathcal{M} \models p(a) \wedge p(b) \text{ and } a \text{ semi-isolates } b$$

by a formula $\theta_u(x, y)$ with $\text{asi-rank } \leq \alpha\}$.

Clearly, $I_p = \text{SI}_{p, 1}$ for any type $p \in S^1(\emptyset)$. Seeing this equality and $\mathfrak{SI}_{\nu(p), 1, 2} = \mathfrak{P}_{\nu(p)}$ the following proposition generalizes Proposition 4.3 in [2].

Proposition 5.13. *Let $p(x)$ be a complete type of a theory T , $\nu(p)$ be a regular labelling function, and α be a positive ordinal. The following conditions are equivalent:*

- (1) *the relation $\text{SI}_{p, \alpha}$ (on a set of realizations of p in any model $\mathcal{M} \models T$) is transitive;*
- (2) *the structure $\mathfrak{SI}_{\nu(p), \alpha}$ is an almost α -deterministic monoid.*

Proof. Let a , b , and c be realizations of p such that $(a, b) \in \text{SI}_{p,\alpha}$ and $(b, c) \in \text{SI}_{p,\alpha}$ by semi-isolating formulas $\theta_{u_1}(a, y)$ and $\theta_{u_2}(b, y)$. If the structure $\mathfrak{SI}_{\nu(p),\alpha}$ is an almost α -deterministic monoid then $\text{si}(u_1 \circ u_2) \leq \alpha$ and the pair (a, c) belongs to $\text{SI}_{p,\alpha}$ by the semi-isolating formula $\theta_{u_1, u_2}(x, y)$. Since elements a , b , and c are arbitrary we have $(2) \Rightarrow (1)$.

Assume now that for some $u_1, u_2 \in \rho_{\nu(p),\alpha}$ the set $\text{SI}_p(u_1, u_2)$ contains a label u such that $\text{si}(u) > \alpha$. Then by compactness the set

$$q(a, y) \Rightarrow \{\theta_{u_1, u_2}(a, y)\} \cup \{\neg\theta_v(a, y) \mid v \in \text{SI}_p(u_1, u_2), \text{si}(v) \leq \alpha\}$$

is consistent, where $\models p(a)$. Consider realizations b and c of p such that $\models \theta_{u_1}(a, b) \wedge \theta_{u_2}(b, c)$ and $\models q(a, c)$. We have $(a, b) \in \text{SI}_{p,\alpha}$ and $(b, c) \in \text{SI}_{p,\alpha}$ but $(a, c) \notin \text{SI}_{p,\alpha}$ by the construction of q . Thus, the relation $\text{SI}_{p,\alpha}$ is not transitive and the implication $(1) \Rightarrow (2)$ holds. \square

Note that for any ordinal $\alpha > 0$ there are no (p, θ_u, p) -edges, linking distinct realizations of p and satisfying the conditions $u > 0$, $\text{si}(u) \leq \alpha$, and $\text{si}(u^{-1}) \leq \alpha$, if and only if the relation $\text{SI}_{p,\alpha}$ is antisymmetric. Since $\text{SI}_{p,\alpha}$ is reflexive, the definition of $\nu(p)$ and Propositions 1.3, 5.13 imply

Corollary 5.14. *Let $p(x)$ be a complete type of a theory T , $\nu(p)$ be a regular labelling function, and α be a positive ordinal. The following conditions are equivalent:*

- (1) *the relation $\text{SI}_{p,\alpha}$ is a partial order on a set of realizations of p in any model $\mathcal{M} \models T$;*
- (2) *the structure $\mathfrak{SI}_{\nu(p),\alpha}$ is an almost α -deterministic monoid and $\rho_{\nu(p),\alpha} \subseteq U^{\leq 0}$.*

This partial order $\mathfrak{SI}_{\nu(p),\alpha}$ is identical if and only if $\rho_{\nu(p),\alpha} = \{0\}$. If $\text{SI}_{p,\alpha}$ is not identical, it has infinite chains.

Propositions 1.3 and 5.13 imply also

Corollary 5.15. *Let $p(x)$ be a complete type of a theory T , $\nu(p)$ be a regular labelling function, and α be a positive ordinal. The following conditions are equivalent:*

- (1) *the relation $\text{SI}_{p,\alpha}$ is an equivalence relation on the set of realizations of p in any model $\mathcal{M} \models T$;*
- (2) *the structure $\mathfrak{SI}_{\nu(p),\alpha}$ is an almost α -deterministic monoid and consists of labels in $U^{\geq 0}$.*

Recall [2] that an element $u \in \rho_{\nu(p)}$ is called (*almost*) *deterministic* if for any/some realization a of p the formula $\theta_u(a, y)$ has unique solution (has finitely many solutions).

Since each semi-isolating formula $\theta_u(a, y)$ with finitely many solutions is equivalent to a disjunction of isolating formulas $\theta_{u_i}(a, y)$, each almost deterministic element has the si-rank 1 and so belongs to the set of labels in the structure $\mathfrak{SI}_{\nu(p), 1, n+1}$, where n is the number of solutions for $\theta_u(a, y)$. In particular, each deterministic element belongs to the set of labels in the structure $\mathfrak{SI}_{\nu(p), 1, 2}$.

It is shown in [2, Proposition 4.7] that if elements u and v are (almost) deterministic then each element v' in $u \cdot v$ is also (almost) deterministic. Hence, the si-rank 1 is preserved for compositions $u \circ v$ of (almost) deterministic elements u and v . Moreover, the si-degree 1 is preserved for compositions of deterministic elements.

In Figure 1, the fragments of Hasse diagram are presented illustrating the links of the structure $\mathfrak{SI} \Leftarrow \mathfrak{SI}_{\nu(p)}$ with structures above, being restrictions of \mathfrak{SI} to subalphabets of U . Here the superscripts $\cdot^{\leq 0}$ and $\cdot^{\geq 0}$ point out on restrictions of \mathfrak{SI} to the sets $U^{\leq 0}$ and $U^{\geq 0}$ respectively, and the subscripts to the upper estimates for si-ranks and si-degrees of labels. In Figure 1, a, a hierarchy of structures \mathfrak{SI}_α , $\alpha \leq \text{si}(p)$, is depicted starting with the trivial substructure; in Figure 1, b, links between substructures of $\mathfrak{SI}_{\nu(p), 1}$ are presented; in Figure 1, c, links between substructures of $\mathfrak{SI}_{\alpha+1}$ for $1 \leq \alpha < \text{si}(p)$ are shown. For a limit ordinal $\beta \leq \text{si}(p)$, the Hasse diagram for substructures of \mathfrak{SI}_β is obtained by union of presented diagrams for $\alpha < \beta$. If an ordinal $\beta \leq \text{si}(p)$ is not limit, the Hasse diagram corresponds to the union of presented diagrams for $\alpha < \beta$ with the removal of structures $\mathfrak{SI}_{\beta+1, 2}^{\leq 0}$ and $\mathfrak{SI}_{\beta+1, 2}^{\geq 0}$.

6. POSTC-monoids

In this Section, we shall consider both the monoids $\mathfrak{SI}_{\nu(p)}$ and their expansions (with the addition of empty set to the universe such that $X \cdot \emptyset = \emptyset \cdot X = \emptyset$ for $X \in \mathcal{P}(\rho_{\nu(p)})$ ¹) by operations and relations of POSTC-algebras containing these monoids. These expansions

$$\mathfrak{M}_{\nu(p)} \Leftarrow \langle \mathcal{P}(\rho_{\nu(p)}); \cdot, \leq, \vee, \wedge, (\cdot \wedge \neg \cdot), \circ \rangle$$

¹This extension forms also a monoid with the unit \emptyset instead of $\{0\}$.

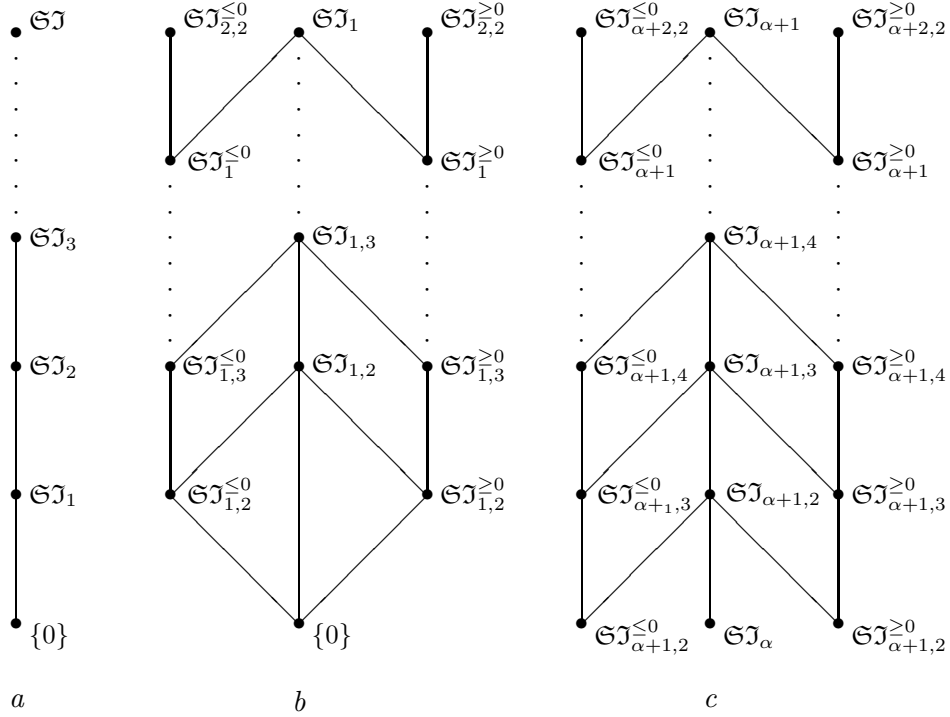


Fig. 1

are called *preordered monoids with relative set-theoretic operations and compositions* over regular labelling functions $\nu(p)$, or briefly $\text{POSTC}_{\nu(p)}$ -monoids.

We collect basic structural properties of $\text{POSTC}_{\nu(p)}$ -monoids and show that any expanded monoid \mathfrak{SI} , satisfying the following list of properties, coincides with some $\text{POSTC}_{\nu(p)}$ -monoid $\mathfrak{M}_{\nu(p)}$.

Let $U = U^- \dot{\cup} \{0\} \dot{\cup} U^+ \dot{\cup} U'$ be an alphabet consisting of a set U^- of *negative elements*, a set U^+ of *positive elements*, a set U' of *neutral elements*, and zero 0. As above, we write $u < 0$ for any element $u \in U^-$, $u > 0$ for any element $u \in U^+$, and $u \cdot v$ instead of $\{u\} \cdot \{v\}$ considering an operation \cdot on the set $\mathcal{P}(U)$; $U^{\leq 0} \equiv U^- \cup \{0\}$, $U^{\geq 0} \equiv U^+ \cup \{0\}$.

A structure $\mathfrak{M} = \langle \mathcal{P}(U); \cdot, \leq, \vee, \wedge, (\cdot \wedge \neg \cdot), \circ \rangle$ is called a *POSTC-monoid* if it satisfies the following conditions:

- the operation \cdot of the monoid $\langle \mathcal{P}(U) \setminus \{\emptyset\}; \cdot \rangle$ with the unit $\{0\}$ is generated by the function \cdot on elements in U such that each elements $u, v \in U$ define a nonempty set $(u \cdot v) \subseteq U$: for any sets $X, Y \in \mathcal{P}(U) \setminus \{\emptyset\}$ the following equality holds:

$$X \cdot Y = \bigcup \{u \cdot v \mid u \in X, v \in Y\};$$

if $X \in \mathcal{P}(U)$ then $X \cdot \emptyset = \emptyset \cdot X = \emptyset$;

the relation \leq on the set $\mathcal{P}(U)$ is a preorder with the least element \emptyset ; this preorder is induced by the partial order \leq' on the set U of labels (forming a upper semilattice) by the following rule: if $X, Y \in \mathcal{P}(U)$ then $X \leq Y$ if and only if $X = \emptyset$, or for any label $u \in X$ there is a label $v \in Y$ with $u \leq' v$ and for any label $v \in Y$ there is a label $u \in X$ with $u \leq' v$;

- a label $u \in U$ is called an *atom* if $v \leq u$ implies $v = u$ for any label $v \in U$; only labels in $U^- \dot{\cup} \{0\} \dot{\cup} U^+$ may be atoms; the label 0 is an atom; some labels in $U^{\geq 0}$ lay under each label in U' , moreover, if only labels $v \in U^{\geq 0}$ lay under a label $u \in U'$ then there is no greatest labels among labels v ; only labels in U' lay over each label in U' ;

- the operations $\vee, \wedge, (\cdot \wedge \neg \cdot)$ on the set $U \cup \{\emptyset\}$ form a distributive lattice with relative complements, moreover, for any elements $u, v \in U \cup \{\emptyset\}$,

$$u \leq' v \Leftrightarrow u \wedge v = u \Leftrightarrow u \vee v = v,$$

$$(u \wedge \neg v) = \emptyset \Leftrightarrow u \leq v;$$

- the operation \circ is defined on the set U such that for any labels $u, v \in U$ the label $u \circ v$ is the greatest element of the set $u \cdot v$;

- the operations \vee, \wedge, \circ on the set $\mathcal{P}(U)$ are induced by the correspondent operations on the set $U \cup \{\emptyset\}$: if $X, Y \in \mathcal{P}(U)$ and $\tau \in \{\vee, \wedge, \circ\}$ then $X \tau Y = \{u \tau v \mid u \in X, v \in Y\}$; the operation $(\cdot \wedge \neg \cdot)$ on the set $\mathcal{P}(U)$ is also induced by the correspondent operation on the set $U \cup \{\emptyset\}$: if $X, Y \in \mathcal{P}(U)$ then $X \wedge \neg Y = \{u \wedge \neg v \mid u \in X, v \in Y\}$;

- the sets $U^- \cup \{\emptyset\}$ and $U^{\geq 0} \cup \{\emptyset\}$ are closed with respect to the operations $\vee, \wedge, (\cdot \wedge \neg \cdot)$; the set U' is closed under the operation \vee ; if $u \in U^-$ and $v \in U^{\geq 0}$ then $(u \vee v) \in U'$;

- repeating the definition in Section 2, for each label $u \in U$, the *rank of semi-isolation* $\text{si}(u) \geq 1$ and the *degree of semi-isolation* $\text{deg}(u)$ of label u is

defined inductively, $\text{si}(\emptyset) = 0$, $\deg(\emptyset) = 1$, as well as equivalence relations \sim_α , restrictions X_α , $X_{\alpha,\beta}$ of sets $X \in \{U, U \cup \{\emptyset\}\}$ and restrictions \mathfrak{M}'_α , $\mathfrak{M}'_{\alpha,\beta}$ for restrictions \mathfrak{M}' of \mathfrak{M} to sets of labels of si-ranks $\leq \alpha$, and for labels of si-rank α to sets of labels of si-degree $< \beta$;

- the restriction $\langle \mathcal{P}(U) \setminus \{\emptyset\}; \cdot \rangle_{1,2}$ of the monoid $\langle \mathcal{P}(U) \setminus \{\emptyset\}; \cdot \rangle$ is a I -groupoid;
- if $u < 0$ then sets $u \cdot v$ and $v \cdot u$ consist of negative elements for any $v \in U$;
- if $u > 0$ and $v > 0$ then $(u \cdot v) \subseteq U^{\geq 0}$;
- if $u, v \in U^{\geq 0} \cup U'$, and $u \in U'$ or $v \in U'$, then $(u \cdot v) \subseteq U'$;
- for any element $u > 0$ there is a nonempty set u^{-1} of *inverse* elements $u' > 0$ such that $0 \in (u \cdot u') \cap (u' \cdot u)$; in this case if $u \leq' v$ and $v \in U^+$ then $u^{-1} \subseteq v^{-1}$;
- if a positive element u belongs to a set $v_1 \cdot v_2$, where $v_1 \circ v_2 \in U^+$, then $u^{-1} \subseteq v_2^{-1} \cdot v_1^{-1}$.

By the definition each POSTC-monoid \mathfrak{M} contains POSTC-submonoids $\mathfrak{M}^{\leq 0}$ and $\mathfrak{M}^{\geq 0}$ with the universes $\mathcal{P}(U^- \cup \{0\})$ and $\mathcal{P}(U^+ \cup \{0\})$ respectively, being also POSTC-monoids (with $U^+ \cup U' = \emptyset$ and $U^- \cup U' = \emptyset$ respectively).

A POSTC-monoid \mathfrak{M} is called *atomic* if for any label $u \in U$ there is an atom $v \in U$ such that $v \leq u$.

Theorem 6.1. *For any (at most countable and having an ordinal $\sup\{\text{si}(u) \mid u \in U\}$) POSTC-monoid \mathfrak{M} there is a (small) theory T with a type $p(x) \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{M}_{\nu(p)} = \mathfrak{M}$.*

Proof follows the same scheme as the proof of [2, Theorem 6.1] and, for the structure $\langle \mathcal{P}(U) \setminus \{\emptyset\}; \cdot \rangle_{1,2}$, it repeats this proof word for word. Since the proof of [2, Theorem 6.1] is voluminous we only point out the distinctive features leading to the proof of this theorem.

1. A binary predicate Q_u is defined for each element $u \in U \cup \{\emptyset\}$. This predicate links only elements of the same colors if $u \geq 0$, and defines a Q_u -ordered coloring Col if $u \in U^- \cup U'$; $Q_\emptyset = \emptyset$.
2. For any elements $u, v \in U \cup \{\emptyset\}$ the following condition is satisfied: $u \leq' v \Leftrightarrow Q_u \subseteq Q_v$.

3. For any elements $u, v \in U \cup \{\emptyset\}$ the following conditions hold:

$$u_1 \vee u_2 = v \Leftrightarrow Q_{u_1} \cup Q_{u_2} = Q_v,$$

$$u_1 \wedge u_2 = v \Leftrightarrow Q_{u_1} \cap Q_{u_2} = Q_v,$$

$$u_1 \wedge \neg u_2 = v \Leftrightarrow Q_{u_1} \setminus Q_{u_2} = Q_v,$$

$$u_1 \circ u_2 = v \Leftrightarrow Q_{u_1} \circ Q_{u_2} = Q_v.$$

In particular, the predicates Q_{u_1} and Q_{u_2} are disjoint if and only if $u_1 \wedge u_2 = \emptyset$. \square

Remark 6.2. Since labels $u \in \rho_{\nu(p)}$ for semi-isolating formulas admit complements in $\rho_{\nu(p)}$ only for principal types p (and these complements are defined relative to the isolating formula of p), unlike I -groupoids, if a POSTC-monoid \mathfrak{M} is constructed by a set $U^{\geq 0}$, it admits a representation in a transitive theory T with a (unique) type $p(x) \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{M}_{\nu(p)} = \mathfrak{M}$ if and only if the set-theoretic operations in \mathfrak{M} form a Boolean algebra.

7. Partial POSTC-monoid on a set of realizations for a family of 1-types of a complete theory

In this section, the results above for a structure of a type, as well as results in [2] for isolating formulas, are generalized for a structure on a set of realizations for a family of types.

Let R be a nonempty family of types in $S^1(T)$. We denote by $\nu(R)$ a regular family of labelling functions

$$\nu(p, q): \text{PF}(p, q)/\text{PE}(p, q) \rightarrow U, \quad p, q \in R,$$

$$\rho_{\nu(R)} \rightleftharpoons \bigcup_{p, q \in R} \rho_{\nu(p, q)}.$$

As in Proposition 4.1, the partial (for $|R| > 1$) function SI on the set $R \times \mathcal{P}(U) \times R$, which maps each tuple of triples $(p_1, u_1, p_2), \dots, (p_k, u_k, p_{k+1})$, where $u_1 \in \rho_{\nu(p_1, p_2)} \cup \{\emptyset\}, \dots, u_k \in \rho_{\nu(p_k, p_{k+1})} \cup \{\emptyset\}$, to the set of triples (p_1, v, p_{k+1}) , where $v \in \text{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1})$, is associative:

$$\begin{aligned} \text{SI}(\text{SI}(p_1, u_1, p_2, u_2, p_3), u_3, p_4) &= \text{SI}(p_1, u_1, p_2, u_2, p_3, u_3, p_4) = \\ &= \text{SI}(p_1, u_1, \text{SI}(p_2, u_2, p_3, u_3, p_4)) \end{aligned} \tag{2}$$

for $u_1 \in \rho_{\nu(p_1, p_2)} \cup \{\emptyset\}$, $u_2 \in \rho_{\nu(p_2, p_3)} \cup \{\emptyset\}$, $u_3 \in \rho_{\nu(p_3, p_4)} \cup \{\emptyset\}$.

Consider the structure

$$\mathfrak{M}_{\nu(R)} \Rightarrow \langle R \times \mathcal{P}(U) \times R; \cdot, \preceq, \vee, \wedge, (\cdot \wedge \neg \cdot), \circ \rangle$$

with the partial operations \cdot and \circ such that

$$(p_1, X_1, p_2) \cdot (p_2, X_2, p_3) = \bigcup \{(p_1, u_1, p_2) \cdot (p_2, u_2, p_3) \mid u_1 \in X_1, u_2 \in X_2\},$$

$$(p_1, u_1, p_2) \cdot (p_2, u_2, p_3) = \{(p_1, v, p_3) \mid v \in \text{SI}(p_1, u_1, p_2, u_2, p_3)\},$$

$$(p_1, X_1, p_2) \circ (p_2, X_2, p_3) = \bigcup \{(p_1, u_1, p_2) \circ (p_2, u_2, p_3) \mid u_1 \in X_1, u_2 \in X_2\},$$

$$(p_1, u_1, p_2) \circ (p_2, u_2, p_3) = \{(p_1, u \circ v, p_3)\},$$

$$u_1 \in \rho_{\nu(p_1, p_2)} \cup \{\emptyset\}, u_2 \in \rho_{\nu(p_2, p_3)} \cup \{\emptyset\},$$

as well as the relation \preceq of preorder, being induced by the partial order, of the same name, on the set of labels and the partial operations $\vee, \wedge, (\cdot \wedge \neg \cdot)$ such that

$$(p, X, q) \vee (p, Y, q) = \bigcup \{(p, u, q) \vee (p, v, q) \mid u \in X, v \in Y\},$$

$$(p, u, q) \vee (p, v, q) = \{(p, u \vee v, q)\},$$

$$(p, X, q) \wedge (p, Y, q) = \bigcup \{(p, u, q) \wedge (p, v, q) \mid u \in X, v \in Y\},$$

$$(p, u, q) \wedge (p, v, q) = \{(p, u \wedge v, q)\},$$

$$(p, X, q) \wedge \neg(p, Y, q) = \bigcup \{(p, u, q) \wedge \neg(p, v, q) \mid u \in X, v \in Y\},$$

$$(p, u, q) \wedge \neg(p, v, q) = \{(p, u \wedge \neg v, q)\},$$

$$u, v \in \rho_{\nu(p, q)} \cup \{\emptyset\}.$$

The POSTC-monoids $\mathfrak{M}_{\nu(p)}$, $p \in R$, are naturally embeddable in this structure. The structure $\mathfrak{M}_{\nu(R)}$ is called a *join of POSTC-monoids* $\mathfrak{M}_{\nu(p)}$, $p \in R$, relative to the family $\nu(R)$ of labelling functions and it is denoted by $\bigoplus_{\nu(R)} \mathfrak{M}_{\nu(p)}$. If $\rho_{\nu(p, q)} = \emptyset$ for all $p \neq q$ the join $\bigoplus_{\nu(R)} \mathfrak{M}_{\nu(p)}$ is *free*, it is represented as the disjoint union of POSTC-monoids $\mathfrak{M}_{\nu(p)}$ and denoted by $\bigsqcup_{p \in R} \mathfrak{M}_{\nu(p)}$.

By (2) we have

Proposition 7.1. *For any complete theory T , for any nonempty family $R \subset S(T)$ of 1-types, and for any regular family $\nu(R)$ of labelling functions, each n -ary partial operation $\text{SI}(p_1, \cdot, p_2, \cdot, p_3 \dots, p_n, \cdot, p_{n+1})$ on the set $\mathcal{P}(U)$ is interpretable by a term of the structure $\bigoplus_{p \in \nu(R)} \mathfrak{M}_{\nu(p)}$ with fixed types $p_1, \dots, p_{n+1} \in R$.*

Denote by $\mathfrak{SI}_{\nu(R)}$ the restriction of $\mathfrak{M}_{\nu(R)} \upharpoonright R \times (\mathcal{P} \setminus \{\emptyset\}) \times R$ to the partial operation \cdot .

Using Proposition 1.3 we obtain the following analogue of Proposition 4.3.

Proposition 7.2. *For any complete theory T , for any nonempty family $R \subset S(T)$ of 1-types, and for any regular family $\nu(R)$ of labelling functions, the restriction of the structure $\mathfrak{SI}_{\nu(R)}$ to the set $U^{\leq 0}$ (respectively $U^{\geq 0}$, $U^{\geq 0} \cup U'$) is closed under the partial operation \cdot .*

By Proposition 7.2 the structure $\mathfrak{SI}_{\nu(R)}$ has substructures $\mathfrak{SI}_{\nu(R)}^{\leq 0}$, $\mathfrak{SI}_{\nu(R)}^{\geq 0}$ and $\mathfrak{SI}_{\nu(R)}^{\geq 0, \text{neu}}$, generated by triples (p, u, q) with $u \leq 0$, $u \geq 0$, and $u \in U^{\geq 0} \cup U'$ respectively, $p, q \in R$. Here, for any triple (p, u, q) in $\mathfrak{SI}_{\nu(R)}^{\geq 0}$ the triple (q, u^{-1}, p) is also attributed to $\mathfrak{SI}_{\nu(R)}^{\geq 0}$.

Replacing for the definition in Section 5 the function $\nu(p)$ to the family $\nu(R)$ of functions we obtain the notions of (α, n) -deterministic, α -deterministic, almost α -deterministic, and (α, ω) -deterministic structures $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$.

Below we formulate a series of assertions that immediately transform from the class of structures $\mathfrak{SI}_{\nu(p)}$ to the class of structures $\mathfrak{SI}_{\nu(R)}$.

Proposition 7.3. (Monotony) *If a structure $\mathfrak{SI}_{\nu(R)} \upharpoonright U_0$ is (almost) α -deterministic and β is a positive ordinal then the structure $(\mathfrak{SI}_{\nu(R)} \upharpoonright U_0) \upharpoonright \beta$ is also (almost) α -deterministic.*

Proposition 7.4. *For a structure $\mathfrak{SI}_{\nu(R)}$ and ordinals α, β , where $\alpha, \beta > 0$, $\beta \in \omega + 1$, the following conditions are equivalent:*

- (1) *the structure $\mathfrak{SI}_{\nu(R)}$ is (α, β) -deterministic;*
- (2) *for any types $p, q, r \in R$ and labels $u_1 \in \rho_{\nu(p,q), \alpha, \beta}$, $u_2 \in \rho_{\nu(q,r), \alpha, \beta}$ the inequality $\text{si}(u_1 \circ u_2) \leq \alpha$ holds and is $\text{si}(u_1 \circ u_2) = \alpha$ then $\deg(u_1 \circ u_2) < \beta$.*

Corollary 7.5. *For a structure $\mathfrak{SI}_{\nu(R)}$ and a positive ordinal α , the following conditions are equivalent:*

- (1) *the structure $\mathfrak{SI}_{\nu(R)}$ is α -deterministic;*

(2) $\text{si}(u_1 \circ u_2) \leq \alpha$ for any types $p, q, r \in R$ and labels $u_1 \in \rho_{\nu(p,q),\alpha}$, $u_2 \in \rho_{\nu(q,r),\alpha}$.

Corollary 7.6. *If $\text{si}(R)$ is an ordinal then the structure $\mathfrak{S}\mathfrak{I}_{\nu(R)}$ is almost $\text{si}(R)$ -deterministic.*

Proposition 7.7. *If a structure $\mathfrak{S}\mathfrak{I}_{\nu(R)}$ is (α, β) -deterministic then $\mathfrak{S}\mathfrak{I}_{\nu(R),\alpha} \Rightarrow \mathfrak{S}\mathfrak{I}_{\nu(R)} \upharpoonright \alpha$ is also (α, β) -deterministic.*

Proposition 7.8. *If $\text{si}(R)$ is an ordinal then the structure $\mathfrak{S}\mathfrak{I}_{\nu(R)}$ is $\text{si}(R)$ -deterministic if and only if the value $\deg(R)$ is not defined or equals 1.*

Proposition 7.9. *A structure $\mathfrak{P}_{\nu(R)}$ is (almost) deterministic if and only if $\mathfrak{S}\mathfrak{I}_{\nu(R),1,2}$ is (almost) 1-deterministic.*

Let R be a nonempty family of complete 1-types of a theory T , $\nu(R)$ be a regular family of labelling functions, and α be an ordinal, $\alpha > 0$. The structure $\mathfrak{S}\mathfrak{I}_{\nu(R)}$ is called *locally α -deterministic* if for any nonempty finite set $R_0 \subseteq R$ there is a natural number $n \geq 2$ such that the structure $\mathfrak{S}\mathfrak{I}_{\nu(R_0)}$ is (α, n) -deterministic.

Repeating the proof of Proposition 5.8 we obtain

Proposition 7.10. *Let R be a nonempty family of complete 1-types of a theory T , $\nu(R)$ be a regular family of labelling functions, $\text{si}(R) < \omega$. The following conditions are equivalent:*

- (1) *the structure $\mathfrak{S}\mathfrak{I}_{\nu(R)}$ is locally 1-deterministic;*
- (2) *the set $\rho_{\nu(p,q)}$ is finite for any $p, q \in R$;*
- (3) *the set $\rho_{\nu(p,q),1}$ is finite for any $p, q \in R$;*
- (4) *the set $\rho_{\nu(p,q),1,2}$ (consisting of all atoms $u \in \rho_{\nu(p,q)}$) is finite for any $p, q \in R$.*

The notion of (n, p) -type is generalized in the following definition.

Definition (K. Ikeda, A. Pillay, A. Tsuboi [20]). Let $p_1(x_1), \dots, p_n(x_n)$ be types in $S(T)$ with disjoint free variables. A type $q(x_1, \dots, x_n) \in S(T)$ is said to be a (p_1, \dots, p_n) -type if $q(x_1, \dots, x_n) \supseteq \bigcup_{i=1}^n p_i(x_i)$. The set of all (p_1, \dots, p_n) -types of T is denoted by $S_{p_1, \dots, p_n}(T)$. A theory T is *almost ω -categorical* if for any types $p_1(x_1), \dots, p_n(x_n) \in S(T)$ there are only finitely many types $q(x_1, \dots, x_n) \in S_{p_1, \dots, p_n}(T)$.

Definition (B. S. Baizhanov, S. V. Sudoplatov, V. V. Verbovskiy [16]). A type $q(\bar{x})$ in $S_{p_1, \dots, p_n}(T)$ is said to be (p_1, \dots, p_n) -principal if there is a

formula $\varphi(\bar{y}) \in q(\bar{x})$ such that

$$\cup\{p_i(x_i) \mid i = 1, \dots, n\} \cup \{\varphi(\bar{x})\} \vdash q(\bar{x}).$$

The following lemma obviously generalizes Lemma 5.10.

Lemma 7.11 [16]. *For any types $p_1(x_1), \dots, p_n(x_n) \in S(\emptyset)$ the following conditions are equivalent:*

- (1) *the set of (p_1, \dots, p_n) -types with free variables in (x_1, \dots, x_n) is finite;*
- (2) *any (p_1, \dots, p_n) -type is (p_1, \dots, p_n) -principal.*

By Lemma 7.11, a theory T is almost ω -categorical if and only if for any types $p_1(x_1), \dots, p_n(x_n) \in S^1(T)$, each (p_1, \dots, p_n) -type is (p_1, \dots, p_n) -principal.

Proposition 7.10 and Lemma 7.11 imply

Corollary 7.12. *If R is a nonempty family of complete 1-types of a theory T , $\nu(R)$ is a regular family of labelling functions, and all (p_1, p_2) -types, where $p_1, p_2 \in R$, are (p_1, p_2) -principal then the structure $\mathfrak{SI}_{\nu(R)}$ is locally 1-deterministic.*

Corollary 7.13. *If T is an almost ω -categorical theory and $\nu(S^1(\emptyset))$ is a regular family of labelling functions then the structure $\mathfrak{SI}_{\nu(R)}$ is locally 1-deterministic.*

For a nonempty family R of 1-types in $S(T)$ and a positive ordinal α , we denote by $\text{SI}_{R,\alpha}$ (in a model \mathcal{M} of T) the restriction of SI_R to the set of formulas of si-ranks $\leq \alpha$:

$$\text{SI}_{R,\alpha} \rightleftharpoons \{(a, b) \mid \text{tp}(a), \text{tp}(b) \in R \text{ and } a \text{ semi-isolates } b$$

$$\text{ by a formula } \theta_{\text{tp}(a), u, \text{tp}(b)}(x, y), \text{ with a si-rank } \leq \alpha\}.$$

Clearly, $I_R = \text{SI}_{R,1}$ for any nonempty family R of 1-types. Considering this equality and the equality $\mathfrak{SI}_{\nu(R),1,2} = \mathfrak{P}_{\nu(R)}$, the following proposition generalizes Proposition 5.13 as well as Propositions 4.3 and 8.3 in [2].

Proposition 7.14. *Let R be a nonempty family of complete 1-types of a theory T , $\nu(R)$ be a regular family of labelling functions, and α be a positive ordinal. The following conditions are equivalent:*

- (1) *the relation $\text{SI}_{R,\alpha}$ (on a set of realizations of types $p \in R$ in any model $\mathcal{M} \models T$) is transitive;*

(2) the structure $\mathfrak{SI}_{\nu(R),\alpha}$ is almost α -deterministic.

Proof repeats the proof of Proposition 5.13 almost word for word. \square

Propositions 1.3 and 7.14 imply the following assertions.

Corollary 7.15. *Let R be a nonempty family of complete 1-types of a theory T , $\nu(R)$ be a regular family of labelling functions, and α be a positive ordinal. The following conditions are equivalent:*

- (1) the relation $\text{SI}_{R,\alpha}$ in any model $\mathcal{M} \models T$ is a partial order;
- (2) the structure $\mathfrak{SI}_{\nu(R),\alpha}$ is almost α -deterministic and $\rho_{\nu(R),\alpha} \subseteq U^{\leq 0}$.

The partial order $\text{SI}_{R,\alpha}$ is identical if and only if $\rho_{\nu(R),\alpha} = \{0\}$. The non-identical partial order $\text{SI}_{R,\alpha}$ has infinite chains if and only if $|\rho_{\nu(p),\alpha}| > 1$ for some type $p \in R$ or there is a sequence p_n , $n \in \omega$, of pairwise distinct types in R such that $|\rho_{\nu(p_n, p_{n+1}),\alpha}| \geq 1$, $n \in \omega$, or $|\rho_{\nu(p_{n+1}, p_n),\alpha}| \geq 1$, $n \in \omega$.

Corollary 7.16. *Let R be a nonempty family of complete 1-types of a theory T , $\nu(R)$ be a regular family of labelling functions, and α be a positive ordinal. The following conditions are equivalent:*

- (1) the relation $\text{SI}_{R,\alpha}$ on a set of realizations of types $p \in R$ in any model $\mathcal{M} \models T$ is an equivalence relation;
- (2) the structure $\mathfrak{SI}_{\nu(R),\alpha}$ is almost α -deterministic and $\rho_{\nu(R),\alpha} \subseteq U^{\geq 0}$.

The results above substantiate that the diagram in Figure 1 admits the transformation replacing the type p by a nonempty family $R \subseteq S^1(\emptyset)$.

8. POSTC $_{\mathcal{R}}$ -structures

Definition. Let \mathcal{R} be a nonempty set,

$$U = U^- \dot{\cup} \{0\} \dot{\cup} U^+ \dot{\cup} U'$$

be an alphabet consisting of a set U^- of *negative elements*, a set U^+ of *positive elements*, a set U' of *neutral elements*, and zero 0. If p and q are elements in \mathcal{R} , we write $u < 0$ and $(p, u, q) < 0$ for any element $u \in U^-$, $u > 0$ and $(p, u, q) > 0$ for any element $u \in U^+$; $U^{\leq 0} \rightleftharpoons U^- \cup \{0\}$, $U^{\geq 0} \rightleftharpoons U^+ \cup \{0\}$. For the set \mathcal{R}^2 of all pairs (p, q) , $p, q \in \mathcal{R}$, we consider a *regular family* $\mu(\mathcal{R})$ of sets $\mu(p, q) \subseteq U$ such that

- $0 \in \mu(p, q)$ if and only if $p = q$;
- $\mu(p, p) \cap \mu(q, q) = \{0\}$ for $p \neq q$;
- $\mu(p, q) \cap \mu(p', q') = \emptyset$ if $p \neq q$ and $(p, q) \neq (p', q')$;

- $\bigcup_{p,q \in \mathcal{R}} \mu(p, q) = U$.

Further we write $\mu(p)$ instead of $\mu(p, p)$, and considering a partial operation \cdot on the set $\mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ we shall write, as above, $(p, u, q) \cdot (q, v, r)$ instead of $(p, \{u\}, q) \cdot (q, \{v\}, r)$.

A structure

$$\mathfrak{M} = \langle \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}; \cdot, \leq, \vee, \wedge, (\cdot \wedge \neg \cdot), \circ \rangle$$

with a regular family $\mu(\mathcal{R})$ of sets is said to be a $\text{POSTC}_{\mathcal{R}}$ -structure if the following conditions hold:

- the partial operation \cdot of the structure $\langle \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}; \cdot \rangle$ has values $(p, X, q) \cdot (p', Y, q')$ only for $p' = q$, $X \subseteq \mu(p, q)$, $Y \subseteq \mu(p', q')$, and it is generated by the function \cdot for elements in U : for any sets $X, Y \in \mathcal{P}(U)$, $\emptyset \neq X \subseteq \mu(p, q)$, $\emptyset \neq Y \subseteq \mu(q, r)$, the following equality is satisfied:

$$(p, X, q) \cdot (q, Y, r) = \bigcup \{(p, x, q) \cdot (q, y, r) \mid x \in X, y \in Y\},$$

and if some of X, Y is empty then $(p, X, q) \cdot (q, Y, r) = \emptyset$;

- each restriction $\mathfrak{M}_{\mu(p)}$ of \mathfrak{M} to $\{p\} \times \mathcal{P}(\mu(p)) \times \{p\}$ is isomorphic to a POSTC -monoid with the universe $\mathcal{P}(\mu(p))$, $p \in \mathcal{R}$; atoms $u \in \mu(p)$ in $\mathfrak{M}_{\mu(p)}$ are called *p-atoms*;

- each restriction $\mathfrak{M}_{\mu(p, q)}$, $p \neq q$, of \mathfrak{M} to $\{p\} \times \mathcal{P}(\mu(p, q)) \times \{q\}$ has empty partial operations \cdot and \circ ; the restriction of $\mathfrak{M}_{\mu(p, q)}$ to the relation \leq is a preordered set $\langle \{p\} \times \mathcal{P}(\mu(p, q)) \times \{q\}; \leq_{p, q} \rangle$ with the least element (p, \emptyset, q) , the preorder $\leq_{p, q}$ of this structure is induced by the partial order $\leq'_{p, q}$ on the set $\mu(p, q)$ of labels (forming a upper semilattice if $\mu(p, q) \neq \emptyset$) by the following rule: if $X, Y \in \mathcal{P}(\mu(p, q))$ then $X \leq_{p, q} Y$ if and only if $X = \emptyset$, or for any label $u \in X$ there is a label $v \in Y$ with $u \leq_{p, q} v$ and for any label $v \in Y$ there is a label $u \in X$ with $u \leq_{p, q} v$;

- a label $u \in \mu(p, q)$, where $p \neq q$, is said to be a (p, q) -atom if $v \leq_{p, q} u$ implies $v = u$ for any label $v \in \mu(p, q)$; only labels in $\mu(p, q) \cap (U^- \cup U^+)$ may be (p, q) -atoms; some labels in $\mu(p, q) \cap U^{\geq 0}$ lay under each label in $\mu(p, q) \cap U'$, moreover, if only labels $v \in \mu(p, q) \cap U^{\geq 0}$ lay under a label $u \in \mu(p, q) \cap U'$ then there are no greatest labels among labels v ; only labels in $\mu(p, q) \cap U'$ lay over each label in $\mu(p, q) \cap U'$;

- the operations $\vee, \wedge, (\cdot \wedge \neg \cdot)$ are defined on each set $\mu(p, q) \cup \{\emptyset\}$ in the structure $\mathfrak{M}_{\mu(p, q)}$ and form a distributive lattice with relative complements on $\mu(p, q) \cup \{\emptyset\}$, moreover, for any elements $u, v \in \mu(p, q) \cup \{\emptyset\}$,

$$u \leq_{p, q} v \Leftrightarrow u \wedge v = u \Leftrightarrow u \vee v = v \Leftrightarrow u \wedge \neg v = \emptyset;$$

- the relation \leq on the set $\mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ is a preorder with minimal elements (p, \emptyset, q) , $p, q \in \mathcal{R}$; this preorder is induced by the union \leq_U of preorders \leq_p in the structures $\mathfrak{M}_{\mu(p)}$, $p \in \mathcal{R}$, and of preorders $\leq_{p, q}$ in the structures $\mathfrak{M}_{\mu(p, q)}$, $p, q \in \mathcal{R}$, $p \neq q$, on sets of labels in these structures: if $X, Y \in \mathcal{P}(U)$ then $(p, X, q) \leq (p', Y, q')$ if and only if $p = p'$, $q = q'$, and $X = \emptyset$ or for any label $u \in X$ there is a label $v \in Y$ with $u \leq_U v$ and for any label $v \in Y$ there is a label $u \in X$ with $u \leq_U v$;

- the partial operations $\vee, \wedge, (\cdot \wedge \neg \cdot)$ are defined on the set $\mathcal{R} \times (U \cup \{\emptyset\}) \times \mathcal{R}$ in the structure \mathfrak{M} being unions of correspondent operations on the sets $\mu(p) \cup \{\emptyset\}$ in $\mathfrak{M}_{\mu(p)}$ and on the sets $\mu(p, q) \cup \{\emptyset\}$ in $\mathfrak{M}_{\mu(p, q)}$, $p \neq q$;

- the partial operation \circ is defined on the set $\mathcal{R} \times (U \cup \{\emptyset\}) \times \mathcal{R}$ in the structure \mathfrak{M} being obtained from the union of correspondent operations in the structures $\mathfrak{M}_{\mu(p)}$, $p \in \mathcal{R}$, by the following extension: if $u_1 \in \mu(p, q)$ and $u_2 \in \mu(q, r)$ then there is unique element $v \in \mu(p, r)$, such that $(p, u_1, q) \circ (q, u_2, r) = (p, v, r)$; this element v is the $\leq_{p, r}$ -greatest label in the set $(p, u_1, q) \cdot (q, u_2, r)$, it is called a *composition* of elements u_1 and u_2 and it is denoted by $u_1 \circ u_2$;

$$(p, u_1, q) \circ (q, \emptyset, r) = (p, \emptyset, q) \cdot (q, u_2, r) = (p, \emptyset, q) \cdot (q, \emptyset, r) = (p, \emptyset, r);$$

- the partial operations \vee, \wedge, \circ on the set $\mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ are induced by the correspondent partial operations on the set $\mathcal{R} \times (U \cup \{\emptyset\}) \times \mathcal{R}$: if $(p, X, q), (p', Y, q') \in \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ and $\tau \in \{\vee, \wedge, \circ\}$ then the value $(p, X, q) \tau (p', Y, q')$ is not defined or it is defined and coincides with the set $\{(p, u, q) \tau (p', v, q') \mid u \in X, v \in Y\}$, in which all values are defined; the partial operation $(\cdot \wedge \neg \cdot)$ on the set $\mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ is also induced by the correspondent partial operation on the set $\mathcal{R} \times (U \cup \{\emptyset\}) \times \mathcal{R}$: if $(p, X, q), (p', Y, q') \in \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ then the value $(p, X, q) \wedge \neg (p', Y, q')$ is defined only for $p = p'$, $q = q'$, $X, Y \subseteq \mu(p, q)$ and it is equal to $\{(p, u, q) \wedge \neg (p, v, q) \mid u \in X, v \in Y\}$;

- each of the sets $U^- \cup \{\emptyset\}$ and $U^{\geq 0} \cup \{\emptyset\}$ is closed under operations $\vee, \wedge, (\cdot \wedge \neg \cdot)$; the set U' is closed under the operation \vee ; if $u \in U^-$ and $v \in U^{\geq 0}$ then $(u \vee v) \in U'$;

- repeating the definition in Section 2, each label $u \in U$ obtains inductively the *rank of semi-isolation* $\text{si}(u) \geq 1$ and the *degree of semi-isolation* $\text{deg}(u)$, $\text{si}(\emptyset) = 0$, $\text{deg}(\emptyset) = 1$, as well as the following attributes are defined: the equivalence relations \sim_α , restrictions X_α and $X_{\alpha,\beta}$ of sets $X \in \{U, U \cup \{\emptyset\}\}$, and restrictions $\mathfrak{M}'_\alpha, \mathfrak{M}'_{\alpha,\beta}$ for restrictions \mathfrak{M}' of the structure \mathfrak{M} to the set of labels of si-ranks $\leq \alpha$, and for labels of si-rank α to the set of labels of si-degree $< \beta$;

- the restriction $\langle \mathcal{R} \times (\mathcal{P}(U) \setminus \{\emptyset\}) \times \mathcal{R}; \cdot \rangle_{1,2}$ of the structure \mathfrak{M} is an $I_{\mathcal{R}}$ -structure;

- if $u \in \mu(p, q)$ and $u < 0$ then the set $(p, u, q) \cdot (q, v, r)$ and $(r, v', p) \cdot (p, u, q)$ consist of negative elements for any $v \in \mu(q, r)$ and $v' \in (r, p)$;

- if $u \in \mu(p, q)$, $v \in \mu(q, r)$, $u > 0$, and $v > 0$, then the set $(p, u, q) \cdot (q, v, r)$ consists of elements in $U^{\geq 0}$;

- if $u \in \mu(p, q) \cap (U^{\geq 0} \cup U')$, $v \in \mu(q, r) \cap (U^{\geq 0} \cup U')$, and $u \in U'$ or $v \in U'$, then $(p, u, q) \cdot (q, v, r) \subseteq U'$;

- for any element $u \in \mu(p, q)$ with $u > 0$ there is a nonempty set u^{-1} of *inverse* elements $u' > 0$ such that $(p, 0, p) \in (p, u, q) \cdot (q, u', p)$ and $(q, 0, q) \in (q, u', p) \cdot (p, u, q)$, moreover, if $u \leq'_{p,q} v$ and $v \in U^+$ then $u^{-1} \subseteq v^{-1}$;

- if an element (p, u, r) , where $u > 0$, belongs to a set $(p, v_1, q) \cdot (q, v_2, r)$, where $v_1 \circ v_2 \in U^+$, then $(r, u^{-1}, p) \subseteq (r, v_2^{-1}, q) \cdot (q, v_1^{-1}, p)$.

By the definition, each $\text{POSTC}_{\mathcal{R}}$ -structure \mathfrak{M} contains $\text{POSTC}_{\mathcal{R}}$ -substructures $\mathfrak{M}^{\leq 0}$ and $\mathfrak{M}^{\geq 0}$ being restrictions of \mathfrak{M} to the sets $U^{\leq 0}$ and $U^{\geq 0}$ respectively.

A $\text{POSTC}_{\mathcal{R}}$ -structure \mathfrak{M} is called *atomic* if for any label $u \in \mu(p)$, $p \in \mathcal{R}$, there is a p -atom $v \in U$ such that $v \leq_p u$, and for any label $u \in \mu(p, q)$, $p, q \in \mathcal{R}$, $p \neq q$, there is a (p, q) -atom $v \in U$ such that $v \leq_{p,q} u$.

Combining the proof of Theorems 6.1 and 9.1 in [2] as well as the proof of Theorem 6.1, we obtain the following theorem.

Theorem 8.1. *For any (at most countable and having an ordinal $\sup\{\text{si}(u) \mid u \in U\}$) $\text{POSTC}_{\mathcal{R}}$ -structure \mathfrak{M} there is a (small) theory T with a family of 1-types $R \subset S(T)$ and a regular family $\nu(R)$ of labelling functions such that $\mathfrak{M}_{\nu(R)} = \mathfrak{M}$.*

In conclusion, we note that, using the operation \cdot^{eq} , the constructions above can be transformed for an arbitrary family of types in $S(T)$.

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